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DISCRETE SYMMETRIC DYNAMICAL SYSTEMS AT THE MAIN RESONANCES WITH APPLICATIONS TO AXI-SYMMETRIC GALAXIES

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A study of two-degrees-of-freedom systems with a potential which is discrete-symmetric (even in one of the position variables) is carried out for the resonance cases 1:2, 1:1, 2:1 and 1:3. To produce both qualitative and quantitative results, we obtain in each resonance case normal forms by higher order averaging procedures. This method is related to Birkhoff normalization and provides us with rigorous asymptotic estimates for the approximate solutions. The normal forms have been used to obtain a classification of possible local and global bifurcations for these dynamical systems. One of the applications here is to describe the two-parameter family of bifurcations obtained by detuning a one-parameter family studied by Braun. In all the resonances discussed an approximate integral of the motion other than the total energy exists, but in the 2:1 and 1:3 resonance cases this degenerates into the partial energy of the z motion. In conclusion some remarks are made on the relation between two-degrees-of-freedom systems and solutions of the collisionless Boltzmann equation. Moreover we are able to make some observations on the Hénon–Heiles problem and certain classical examples of potentials.

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1. Introduction

Both in (astro-)physics and in mathematics many studies have been devoted to Hamiltonian systems with two degrees of freedom. Mainly through the work of Birkhoff (1927) it has been realized that a certain resonance parameter, in this paper called ω (§4), plays a crucial part in determining the topology of the phase-space as induced by the Hamiltonian. Here we shall consider Hamiltonian systems at the so-called main resonances with a potential, which is discrete-symmetric in one of the variables. Many numerical studies were produced for such systems, for instance by Hénon & Heiles (1964) and by Contopoulos (1967). On the other hand a number of general results are known, obtained by rather abstract methods in bifurcation theory, for instance by Arnold (1963) and by Kummer (1976).

The aim of this paper is to fill in the gap between abstract and concrete results by using rigorous methods from the theory of asymptotic expansions, based on obtaining Birkhoff normal forms by averaging procedures. The advantage of this method is that it gives general insight into the quantitative behaviour of the phase-flow, which can hardly be achieved by numerical integrations of systems with particular potentials. Moreover, it provides us at the same time with qualitative results concerning the existence and stability of the bifurcations which have been found. In each case the precision of the quantitative results can be improved by calculating higher-order asymptotic approximations; for the language of asymptotics employed in this paper, we refer to Verhulst (1975).

Another important aspect is the relation of the results obtained here for dynamical systems with two degrees of freedom (corresponding with a phase-space of dimension four) with continuous systems described by partial differential equations. In the case of galaxies whose dynamical behaviour is governed by the collective gravitational field, this relation can be indicated as follows.

If H is the Hamiltonian determining the motion of each individual particle, q_i ($i = 1, 2, 3$) are three spatial coordinates and p_i are the corresponding three momenta, the distribution function $f(t, p_i, q_i)$ is determined by the Liouville equation

$$\frac{\partial f}{\partial t}+\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}=0. \tag{1}$$

For self-gravitating systems this equation is supplemented by the Poisson equation

$$\Delta U_1 = \rho \tag{2}$$

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in which U_1 is the collective gravitational potential and ρ is the particle density. Equation (1) is sometimes referred to as the collisionless Boltzmann equation, the system of equations (1), (2) is sometimes called the system of Vlasov equations. The relation with studies of a finite dimensional phase-space was pointed out by Jeans (1916), who observed that the distribution function f is a function of the independent integrals of the Lagrangian subsidiary equations (cf. equation (3) in §2). Jeans's paper was the beginning of a large number of studies involving the so-called 'third integral of the galaxy'; for a survey and further references see Ollongren (1962). The mathematical formulation of our problem is given in §2. In §§ 3 and 5 we formulate the problem in the framework of asymptotics and we summarize the theorems which we need. The resonance parameter ω, which is discussed in §4, will be considered in the neighbourhood of the main resonances $\frac{1}{2}$, 1, 2 and 3; other values of ω will be considered in a subsequent paper. It turns out that near the resonances $\frac{1}{2}$ and 1 a 'third integral of the galaxy' exists in a certain asymptotic sense (§§6.2 and 8.2), producing a periodic exchange of energy between the two degrees of freedom.

The situation is different at resonances 2 and 3 owing to certain degenerations of the normal forms (§11). The concept of stability is used with two different meanings. First we consider the orbital stability of individual orbits in a given potential field (§§7.2 and 9.3); secondly we consider the structural stability of our results with respect to perturbations in a set of permissible potentials (§12). Finally in §13 we give a number of preliminary applications and conclusions.

2. The Hamiltonian and the equations of motion

The equations of motion for a star of unit mass are the Lagrangian equations describing the characteristics of the collision-free Boltzmann equation. In cylindrical coordinates r, θ, z they read

$$\left.\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{\partial U_1}{\partial r}, \\ r\ddot{\theta} &= -2\dot{r}\dot{\theta} - \frac{1}{r}\frac{\partial U_1}{\partial \theta}, \\ \ddot{z} &= -\frac{\partial U_1}{\partial z}. \end{aligned}\right\} \tag{3}$$

The potential U_1 is supposed to be axi-symmetric, i.e. $\partial U_1/\partial\theta = 0$, and analytic with respect to r and z in the domains considered. The requirement of analyticity is for the sake of convenience and can be relaxed if necessary. The assumption of axi-symmetry reduces our problem to the case of a dynamical system with two degrees of freedom as we can integrate the second equation to give

$$r^2\dot{\theta} = J$$

in which J is a constant of motion, the angular momentum integral. Following Ollongren (1962) we introduce the reduced potential $U_2(r, z)$ by

$$U_2 = U_1 + \tfrac{1}{2}J^2/r^2.$$

The remaining two equations of motion become

$$\ddot{r} = -\partial U_2/\partial r; \quad \ddot{z} = -\partial U_2/\partial z. \tag{4}$$

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Apart from axi-symmetry we make a second assumption: the potential U_2 (or U_1) is supposed to be discrete-symmetric in z. This implies $U_2(r, z) = U_2(r, -z)$. In an axi-symmetric system, which is discrete-symmetric in z, circular orbits are found as the stationary solutions of equation (4) which correspond with critical points $z = 0$ and $r = r_0$ given by

$$\frac{J^2}{r_0^3} = \frac{\partial U_1}{\partial r}(r_0, 0).$$

We translate $x = r - r_0$ to obtain the equations of motion in the final form

$$\ddot{x} = -\partial U_3/\partial x, \quad \ddot{z} = -\partial U_3/\partial z, \tag{5}$$

in which consequently U_3 is supposed to be analytic with respect to x and z and discrete-symmetric in z; we put

$$U_3 = U_3(x, z^2) = U_2(x + r_0, z).$$

$x = 0$, $z = 0$ is a (non-degenerate) critical point of U_3; at the same time $x = z = \dot{x} = \dot{z} = 0$ is a non-degenerate critical point of the Hamiltonian function

$$h = \tfrac{1}{2}(\dot{x}^2 + \dot{z}^2) + U_3(x, z^2). \tag{6}$$

The Hamiltonian h is the energy integral of system (5) and so, together with the angular momentum integral J, we have two integrals of motion of system (3). These two integrals may be used to solve the collisionless Boltzmann equation (1) as each differentiable function of h and J satisfies this equation with the given restrictions on the potential. However, (5) constitutes a system of equations of the 4th order and other integrals of the system may exist besides the energy integral. Moreover, distribution functions based on only energy and angular momentum produce results which are at variance with observations (for a discussion of these observational results see Ollongren 1962).

We should remark here that the problem of the existence of an integral of motion, independent of energy and angular momentum (a second integral in a two-degrees of freedom system) also plays a natural part in the ergodic theory of dynamical systems in classical mechanics. All these considerations triggered off a large number of studies, both analytic and numerical, on systems with two degrees of freedom. The analytical results on the third integral in the literature thus far concern formal expansions, i.e. expansions which were given without any proof of convergence or asymptotic validity. Extensive numerical results were produced by Ollongren (1962) and by Martinet & Mayer (1973, 1975); particular interesting cases were studied numerically by Hénon & Heiles (1964) and Contopoulos *et al.* (see for instance Contopoulos 1967).

3. Local analysis

An important part of our considerations is connected with the Taylor expansion of $U_3(x, z^2)$ with respect to its arguments:

$$U_3(x, z^2) = \tfrac{1}{2}(x^2 + \omega^2 z^2) - \left(\frac{a_1}{3}x^3 + a_2 x z^2\right) - \left(\frac{b_1}{4}x^4 + \frac{b_2}{2}x^2 z^2 + \frac{b_3}{4}z^4\right) + \dots. \tag{7}$$

The corresponding equations of motion (5) become

$$\ddot{x} + x = (a_1 x^2 + a_2 z^2) + (b_1 x^3 + b_2 x z^2) + \dots,$$
$$\ddot{z} + \omega^2 z = 2a_2 x z + (b_2 x^2 z + b_3 z^3) + \dots.$$

The coefficients in the expansion of the potential are all real constants; we require that $\omega \geqslant \frac{1}{2}$. The choice of the coefficient $\frac{1}{2}$ for x^2 does not signify any restriction of generality as this can be

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achieved by appropriate re-scaling of U_3. In the Hénon–Heiles problem (Hénon & Heiles 1964) we have $\omega = 1$, $a_1 = 1$, $a_2 = -1$, while all higher order coefficients vanish.

To choose the coefficient of z^2 to be $\geqslant \frac{1}{8}$ is of course a restriction, with the consequence that the origin of phase-space $x = \dot{x} = z = \dot{z} = 0$ is a Lyapunov-stable solution of the equations of motion (actually, positivity of the coefficient would suffice for that). This restriction on the coefficient of z^2 is, as the assumption of discrete symmetry in z, motivated by its relevance for the study of orbits in rotating galaxies (for examples of model potentials, see the applications in §13). The Lyapunov stability of the origin of phase-space implies for such a galaxy the Lyapunov stability of the circular orbits around the centre of the galaxy in the plane of symmetry. At the same time it provides the justification of the following local analysis: we re-scale $x = \epsilon\bar{x}$ and $z = \epsilon\bar{z}$, where ϵ is a small positive parameter. The equations of motion become (we drop the bars after transformation)

$$\left.\begin{aligned}\ddot{x}+x &= \epsilon(a_1x^2+a_2z^2)+\epsilon^2(b_1x^3+b_2xz^2)+O(\epsilon^3),\\ \ddot{z}+\omega^2z &= \epsilon 2a_2xz+\epsilon^2(b_2x^2z+b_3z^3)+O(\epsilon^3).\end{aligned}\right\} \tag{8}$$

We wish to study the phase flow induced by the Hamiltonian (6) into $\mathbb{R}^4$ and the behaviour of the orbits in the extended phase-space (behaviour with time). This is achieved by obtaining local results from a study of equations (8).

4. The resonance parameter ω

We may visualize system (8) as a perturbed linear system with, putting $\epsilon = 0$, frequencies 1 and ω. Whether the ratio of these basic frequencies is rational or irrational plays an important part in the theory of normal forms for Hamiltonian systems and its consequences are developed by Birkhoff, Siegel, Moser and others (for a summary see Moser 1973). In a continuous system as a galaxy we have to consider a continuous set of frequency ratios (here $\omega \geqslant \frac{1}{2}$) and one might think of discarding the rationals as they form a set of measure zero in the permitted range. For the irrational frequency ratios one can find formal canonical transformations which solve system (8) (Birkhoff 1927). However, in this approach one is faced with the problem of the divergence of the series introduced in the canonical transformations which is connected with the problem of small denominators (Siegel 1954). This is why in our approach the starting point will be the resonant structure of system (8) which is obtained by starting with the rationals and then admitting the irrationals by small perturbations of the rationals. We put

$$\omega^2 = n^2[1+\delta(\epsilon)], \tag{9}$$

where $n \geqslant \frac{1}{2}$, $n \in Q$; $\delta(\epsilon)$ is a continuous function of the small parameter ϵ, $\delta(\epsilon) = o(1)$. This approach still does not look very systematic as the rationals n form a dense subset of the set $\omega \geqslant \frac{1}{2}$. However, the theory of Birkhoff normal forms provides us with a natural hierarchy of the rationals for $n \geqslant \frac{1}{2}$:

first order resonance if $n = 2$ or $n = \frac{1}{2}$,

second order resonance if $n = 1$ or $n = 3$,

higher order resonances if $n \neq 2, \frac{1}{2}, 1, 3$;

(if we consider the set $n > 0$, we have to include $n = \frac{1}{3}$ as a second order resonance). At the same time the choice of n limits the choice of the order function $\delta(\epsilon)$. This has been made clear by the work of Van der Burgh (1974).

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In this paper we treat the main resonances (first and second order) and we shall show that we have to change the natural hierarchy owing to the occurrence of certain degenerations in the normal forms.

5. Averaging and the modified Birkhoff transformation

To obtain a system of differential equations in the standard form for averaging we transform in the usual way (generalized Van der Pol substitution) to amplitude-phase variables:

$$x = A\cos(t+\phi), \qquad z = B\cos(nt+\psi),$$

$$\dot{x} = -A\sin(t+\phi), \quad \dot{z} = -nB\sin(nt+\psi).$$

Transformation of equation (8) produces with equation (9) the variational equations

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\epsilon\sin(t+\phi)\,[a_1A^2\cos^2(t+\phi)+a_2B^2\cos^2(nt+\psi)]$$
$$-\epsilon^2\sin(t+\phi)\,[b_1A^3\cos^3(t+\phi)+b_2AB^2\cos(t+\phi)\cos^2(nt+\psi)]+O(\epsilon^3), \qquad (10a)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\epsilon\frac{\cos(t+\phi)}{A}[a_1A^2\cos^2(t+\phi)+a_2B^2\cos^2(nt+\psi)]$$
$$-\epsilon^2\cos(t+\phi)\,[b_1A^2\cos^3(t+\phi)+b_2B^2\cos(t+\phi)\cos^2(nt+\psi)]+O(\epsilon^3), \qquad (10b)$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \delta(\epsilon)\frac{n}{2}B\sin(2nt+2\psi)-\epsilon\frac{1}{n}a_2AB\cos(t+\phi)\sin(2nt+2\psi)$$
$$-\frac{\epsilon^2}{n}\sin(nt+\psi)\,[b_2A^2B\cos^2(t+\phi)\cos(nt+\psi)+b_3B^3\cos^3(nt+\psi)]+O(\epsilon^3), \qquad (10c)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \delta(\epsilon)\,n\cos^2(nt+\psi)-\epsilon\frac{2}{n}a_2A\cos(t+\phi)\cos^2(nt+\psi)$$
$$-\frac{\epsilon^2}{n}[b_2A^2\cos^2(t+\phi)\cos^2(nt+\psi)+b_3B^2\cos^4(nt+\psi)]+O(\epsilon^3), \qquad (10d)$$

to which appropriate initial values have to be added. If one wishes to consider values of the amplitude A which are of the order of the small parameter ϵ a somewhat different set of variables is used to avoid the small denominator in equation (10b) (cf. §7).

Averaging of the $O(\delta(\epsilon))$ and the $O(\epsilon)$ terms in the right-hand sides of equations (10) while keeping A, B, ϕ and ψ fixed produces, if $n \neq \frac{1}{2}$,

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{B}}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} = \frac{n}{2}\delta(\epsilon).$$

The tilde is introduced to indicate that we have omitted the $O(\epsilon^2)$ terms. This result is unexpected only for $n = 2$ as in that case the result means that we have no first-order resonance at $n = 2$. If we choose $\delta(\epsilon) = O(\epsilon^2)$ and $n \neq \frac{1}{2}$, the result means quantitatively that by using the Krylov, Bogoliubov and Mitropolsky averaging theorem (Bogoliubov & Mitropolsky 1961) the amplitudes and phases are approximated by their initial values within an error of $O(\epsilon)$ on the time scale $1/\epsilon$. The dynamical system behaves if $n \neq \frac{1}{2}$ like a linear system on the time scale $1/\epsilon$. Clearly, if $n \neq \frac{1}{2}$, significant changes in the dynamical system take place on a longer time scale. If $n = \frac{1}{2}$ we have a first-order resonance; this case is discussed in §§6 and 7.

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It is clear that in a large number of cases, we have to study the behaviour of the phase-flow on a longer time scale than $1/\epsilon$. This is the motivation for introducing a modified Birkhoff transformation according to Van der Burgh (1974). In the case of conservative dynamical systems as considered here, the modified transformation of obtaining normal forms through averaging produces the same results as the classical Birkhoff transformation.

Consider the initial value problem in $\mathbb{R}^n$

$$\frac{dx}{dt} = \epsilon f_1(x,t) + \epsilon^2 f_2(x,t), \quad x(0) = x_0$$

with $f(x,t)$ T-periodic in t and
$$\int_0^T f_1(x,t)\,dt = 0. \tag{11}$$

Introduce the modified Birkhoff transformation

$$x(t) = y(t) + \epsilon \int_0^t f_1(y(t),s)\,ds \tag{12}$$

and one obtains the equation

$$\frac{dy}{dt} = \epsilon^2 g(y,t) + O(\epsilon^3), \quad y(0) = x_0.$$

One can show that the averaging theorem applies to this equation and that for the first asymptotic approximation $\tilde{y}(t)$ of $y(t)$ the following estimate holds

$$x(t) - \tilde{y}(t) = O(\epsilon) \quad \text{on the time scale } 1/\epsilon^2.$$

We found that condition (11) is satisfied for equation (10) if $n \neq \frac{1}{2}$. We shall introduce in these cases transformation (12) and by averaging obtain asymptotic approximations of the amplitudes and phases on the time scale $1/\epsilon^2$. Second-order resonances may occur for $n = 1, 2, 3$.

6. The first-order resonance case $n = \frac{1}{2}$

6.1. *First-order averaging*

To conform with the formulation in most of the literature we rescale the time $t = \bar{t}/\omega$; the equations of motion (8) become (after dropping the bar)

$$\ddot{x} + \omega^{-2} x = \epsilon(a_1 x^2 + a_2 z^2) + \epsilon^2(b_1 x^3 + b_2 xz^2) + O(\epsilon^3),$$
$$\ddot{z} + z = \epsilon 2a_2 xz + \epsilon^2(b_2 x^2 z + b_3 z^3) + O(\epsilon^3)$$

(all the coefficients a_1, a_2, etc., have to be multiplied by ω^{-2} but we absorb this constant in the coefficients). Studying the resonance case $n = \frac{1}{2}$ and allowing for small detuning of the resonance frequency we may put

$$\omega^{-2} = 4[1 + \delta(\epsilon)], \quad \delta(\epsilon) = O(\epsilon).$$

Transformation to amplitude-phase variables

$$x = A\cos(2t+\phi), \qquad z = B\cos(t+\psi),$$
$$\dot{x} = -2A\sin(2t+\phi), \quad \dot{z} = -B\sin(t+\psi),$$

produces the variational equations

$$\dot{A} = \delta(\epsilon)\, A \sin(4t+2\phi) - \tfrac{1}{2}\epsilon \sin(2t+\phi)\,[a_1 A^2 \cos^2(2t+\phi) + a_2 B^2 \cos^2(t+\psi)]$$
$$-\tfrac{1}{2}\epsilon^2 \sin(2t+\phi)\,[b_1 A^3 \cos^3(2t+\phi) + b_2 AB^2 \cos(2t+\phi)\cos^2(t+\psi)] + O(\epsilon^3), \tag{13a}$$

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$$\dot{\phi} = \delta(\epsilon) + \delta(\epsilon)\cos(4t+2\phi) - \frac{\epsilon}{2A}\cos(2t+\phi)\,[a_1 A^2\cos^2(2t+\phi) + a_2 B^2\cos^2(t+\psi)]$$
$$-\tfrac{1}{2}\epsilon^2\cos(2t+\phi)\,[b_1 A^2\cos^3(2t+\phi) + b_2 B^2\cos(2t+\phi)\cos^2(t+\psi)] + O(\epsilon^3), \tag{13b}$$
$$\dot{B} = -\epsilon 2a_2 AB\sin(t+\psi)\cos(t+\psi)\cos(2t+\phi)$$
$$-\epsilon^2\sin(t+\psi)\,[b_2 A^2 B\cos^2(2t+\phi)\cos(t+\psi) + b_3\cos^3(t+\psi)] + O(\epsilon^3), \tag{13c}$$
$$\dot{\psi} = -\epsilon 2a_2 A\cos(2t+\phi)\cos^2(t+\psi) - \epsilon^2\cos(t+\psi)$$
$$\times[b_2 A^2\cos^2(2t+\phi)\cos(t+\psi) + b_3 B^2\cos^3(t+\psi)] + O(\epsilon^3). \tag{13d}$$

Averaging of equations (13*a*–*d*) and dropping the terms of $O(\epsilon^2)$ produces for the first asymptotic approximations $\tilde{A}$, $\tilde{\phi}$, $\tilde{B}$, $\tilde{\psi}$ of the amplitudes and phases

$$\left.\begin{aligned} \frac{d\tilde{A}}{d\tau} &= -\frac{a_2}{8}\tilde{B}^2\sin\tilde{X}, & \frac{d\tilde{B}}{d\tau} &= \frac{a_2}{2}\tilde{A}\tilde{B}\sin\tilde{X},\\ \frac{d\tilde{\phi}}{d\tau} &= \frac{\delta(\epsilon)}{\epsilon} - \frac{a_2}{8}\frac{\tilde{B}^2}{\tilde{A}}\cos\tilde{X}, & \frac{d\tilde{\psi}}{d\tau} &= -\frac{a_2}{2}\tilde{A}\cos\tilde{X}. \end{aligned}\right\} \tag{14}$$

Here, we introduced the time-like variable $\tau = \epsilon t$ and the auxiliary angular variable X by

$$X = \phi - 2\psi.$$

The initial conditions are the same as those imposed on A, ϕ, B and ψ. For the solutions of equation (14) we have the estimates

$$A(t) - \tilde{A}(\epsilon t) = O(\epsilon)$$

on the time-scale $1/\epsilon$, etc. The equations for $\tilde{\phi}$ and $\tilde{\psi}$ can be used to obtain an equation for the approximate phase difference $\tilde{X}$

$$\frac{d\tilde{X}}{d\tau} = \frac{\delta(\epsilon)}{\epsilon} + a_2\left(\tilde{A} - \frac{\tilde{B}^2}{8\tilde{A}}\right)\cos\tilde{X}. \tag{14a}$$

If $a_2 = 0$ equations (14) degenerate and we treat this case separately.

6.2. *Integrals of motion*

As expected, the equations for $\tilde{A}$ and $\tilde{B}$ can be integrated to produce the approximate energy integral

$$4\tilde{A}^2(\tau) + \tilde{B}^2(\tau) = 2E_0, \tag{15}$$

where E_0 is a constant determined by the initial conditions. It is clear that

$$4A^2(t) + B^2(t) - 2E_0 = O(\epsilon) \quad \text{on the time scale } 1/\epsilon.$$

Of course we have a much stronger result; the boundedness of the solutions on the energy manifold (ϵ is small) enables us to conclude that this estimate holds uniformly for $t > 0$.

A second approximate integral of motion can be obtained by eliminating $\tilde{B}$ from the equations for $\tilde{A}$ and $\tilde{X}$ with the aid of equation (15) and integrating the resulting equations. One finds the integral of equation (14)

$$a_2\tilde{A}(E_0 - 2\tilde{A}^2)\cos\tilde{X} - 2\frac{\delta(\epsilon)}{\epsilon}\tilde{A}^2 = I_3 \tag{16a}$$

or with equation (15)

$$\tfrac{1}{2}a_2\tilde{A}\tilde{B}^2\cos\tilde{X} - 2\frac{\delta(\epsilon)}{\epsilon}\tilde{A}^2 = I_3. \tag{16b}$$

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The constant I_3 is determined by the initial conditions. Transforming to cartesian coordinates and after some rearrangements we find that the integral (16*b*) becomes (we omit the tildes)

$$\tfrac{1}{2}a_2(xz^2 - x\dot{z}^2 + 2z\dot{x}\dot{z}) - 2\frac{\delta(\epsilon)}{\epsilon}(x^2 + \tfrac{1}{4}\dot{x}^2) = I_3. \tag{16c}$$

The estimate of asymptotic validity is

$$\tfrac{1}{2}a_2 A(t)\, B^2(t) \cos X(t) - 2\frac{\delta(\epsilon)}{\epsilon}A^2(t) - I_3 = O(\epsilon)$$

on the time scale $1/\epsilon$.

The results obtained in this section for this first-order resonance case are contained in the literature. For an indication of the literature on quantitative results see Van der Burgh (1974); a discussion of the topology of phase-space for Hamiltonian systems at the 2:1 resonance is given by Cushman (1975).

7. Existence and stability of bifurcations in the case $n = \frac{1}{2}$

In the limiting case $\epsilon = 0$ the equations of motion (8) are linear (the potential and the Hamiltonian are quadratic) and in the case $n = \frac{1}{2}$ all solutions are periodic. We are interested in the number of periodic solutions which branch off in the nonlinear problem under the perturbations of the quadratic potential. We look for these periodic solutions for fixed but arbitrary values of the energy (arbitrary within the scope of asymptotic analysis in the neighbourhood of the origin of phase-space). For each value of the energy we expect to find a finite number of periodic solutions, at least two according to a theorem by Weinstein (1973). The periodic solutions thus found will be called local bifurcations with respect to the energy. Usually one looks for near-normal mode solutions by considering small or zero values of amplitude A and B respectively. If however, $a_2 \neq 0$, equation (13*b*) contains a small denominator in the case $A \to 0$. To study the possibility of a near-normal mode solution we regularize the variational equations in the following way. Instead of phase-amplitude variables introduce

$$x = p_1 \cos 2t + q_1 \sin 2t, \qquad z = p_2 \cos t + q_2 \sin t,$$
$$\dot{x} = -2p_1 \sin 2t + 2q_1 \cos 2t, \quad \dot{z} = -p_2 \sin t + q_2 \cos t.$$

Again we construct the variational equations, now for the variables $p_1(t), \ldots, q_2(t)$, by using the method of variation of parameters. We obtain equations analogous to equations (13*a*–*d*), however, without small denominators. Averaging and omitting terms of $O(\epsilon^2)$ yields ($\tau = \epsilon t$)

$$\frac{d\tilde{p}_1}{d\tau} = \frac{\delta(\epsilon)}{\epsilon}\tilde{q}_1 - \frac{a_2}{4}\tilde{p}_2\tilde{q}_2, \tag{17a}$$

$$\frac{d\tilde{q}_1}{d\tau} = -\frac{\delta(\epsilon)}{\epsilon}\tilde{p}_1 + \frac{a_2}{8}(\tilde{p}_2^2 - \tilde{q}_2^2), \tag{17b}$$

$$\frac{d\tilde{p}_2}{d\tau} = \frac{a_2}{2}(\tilde{p}_1\tilde{q}_2 - \tilde{q}_1\tilde{p}_2), \tag{17c}$$

$$\frac{d\tilde{q}_2}{d\tau} = \frac{a_2}{2}(\tilde{p}_1\tilde{p}_2 + \tilde{q}_1\tilde{q}_2). \tag{17d}$$

Putting the amplitude $\tilde{A} = 0$ (normal mode in the z direction) implies $\tilde{p}_1 = \tilde{q}_1 = 0$. Inspection of equations (17*a*, *b*) reveals that this does not produce a solution if $a_2 \neq 0$, except in the trivial

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case of zero energy. The periodic solution corresponding with a normal mode in the x-direction ($\tilde{B} = 0$ or $\tilde{p}_2 = \tilde{q}_2 = 0$) satisfies both equations (14) and (17). This solution can also be found as an exact solution (see §9.2).

We list the periodic solutions found for each value of the energy in the case $a_2 \neq 0$.

7.1. *Existence of local bifurcations with respect to the energy*

Type I: One normal mode solution in the x direction ($\tilde{B} = 0$); the solution exists for all values of $\delta(\epsilon)$ and a_2, provided that the energy manifold remains compact.

The approximate solution is given by

$$\left.\begin{aligned} \tilde{A}(\epsilon t) &= (E_0/2)^{\frac{1}{2}} \\ \tilde{\phi}(\epsilon t) &= \phi(0) + \delta(\epsilon)\, t. \end{aligned}\right\} \tag{18}$$

The corresponding approximate solution for $x(t)$ reads

$$\tilde{x}(\epsilon t) = (E_0/2)^{\frac{1}{2}} \cos\left[\phi(0) + 2t + \delta(\epsilon)\, t\right],$$

where $\tilde{x}(\epsilon t) - x(t) = O(\epsilon)$ on the time scale $1/\epsilon$.

Type II: one periodic solution if $\tilde{X} = 0$. The amplitudes and phases can be derived from equations (14) and (15). We introduce the parameter

$$d = \frac{\delta(\epsilon)}{a_2 \epsilon E_0^{\frac{1}{2}}}$$

and find

$$\left.\begin{aligned} \tilde{A}(\epsilon t) &= \tfrac{1}{3} E_0^{\frac{1}{2}} \left[-d + (d^2 + \tfrac{3}{2})^{\frac{1}{2}}\right] \\ \tilde{B}(\epsilon t) &= \tfrac{2}{3} E_0^{\frac{1}{2}} \left[3 - 2d^2 + 2d(d^2 + \tfrac{3}{2})^{\frac{1}{2}}\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{19}$$

Expressions for $\tilde{\phi}(\epsilon t)$ and $\tilde{\psi}(\epsilon t)$ can be obtained by substitution of the expressions for $\tilde{A}$ and $\tilde{B}$ into equation (14) and integrating. The condition of existence of the type II periodic solution is obtained from the energy integral (15). Clearly we have $0 < A^2 < \frac{1}{2}E_0$; together with equation (19) this gives the condition of existence for the type II bifurgation

$$d > -\tfrac{1}{2}\sqrt{2}. \tag{20}$$

At the value $d = -\frac{1}{2}\sqrt{2}$ the type II periodic solution branches off the type I (normal mode) solution.

Type III: one periodic solution if $\tilde{X} = \pi$. The amplitudes and phases can be derived from equations (14) and (15). We find

$$\left.\begin{aligned} \tilde{A}(\epsilon t) &= \tfrac{1}{3} E_0^{\frac{1}{2}} \left[d + (d^2 + \tfrac{3}{2})^{\frac{1}{2}}\right], \\ \tilde{B}(\epsilon t) &= \tfrac{2}{3} E_0^{\frac{1}{2}} \left[3 - 2d^2 - 2d(d^2 + \tfrac{3}{2})^{\frac{1}{2}}\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{21}$$

Again expressions for $\tilde{\phi}(\epsilon t)$ and $\tilde{\psi}(\epsilon t)$ can be obtained easily. Both for the type II and the type III bifurcation the resulting $\tilde{x}(\epsilon t)$ and $\tilde{z}(\epsilon t)$ constitute approximate solutions with error $O(\epsilon)$ on the time scale $1/\epsilon$. The condition of existence for the type III bifurcation is obtained from the energy integral and becomes

$$d < \tfrac{1}{2}\sqrt{2}. \tag{22}$$

At the value $d = \frac{1}{2}\sqrt{2}$ the type III periodic solution branches off the type I (normal mode) solution.

Remarks

(1) The parameter d depends on the energy E_0 and this is connected with the fact that the effectivity of the detuning $\delta(\epsilon)$ depends on the energy. If, for example, we fix $\delta(\epsilon)/(a_2\epsilon) = 1$, for

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small values of the energy the detuning suppresses the type III bifurcation; type I and type II bifurcations exist. If the energy is increased the phase flow remains qualitatively unchanged until at $E_0 = 2$ the existence of the type III bifurcation is prompted. For $E_0 > 2$ all three bifurcations exist. It should be remarked here that these inferences lose their validity beyond values of E_0 which cause the energy manifold to be non-compact.

(2) In numerical computations one studies periodic solutions and the phase-flow by a surface of section. For instance, for fixed values of the energy and $z = 0, \dot{z} > 0$, one pictures the behaviour in the x, $\dot{x}$ plane (see, for example, Hénon & Heiles 1964). It is clear that the set of fixed points in general does not present a complete picture of the periodic solutions. For instance in the case of the bifurcations treated here, the boundary in a x, $\dot{x}$ plane ($z = 0$) represents another periodic solution; the boundary in a z, $\dot{z}$ plane ($x = 0$) represents no periodic solution.

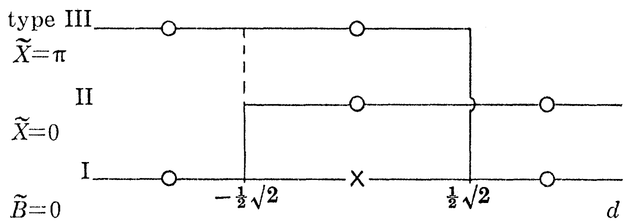

FIGURE 1. The existence of bifurcations (periodic solutions) for each value of the (small) energy in a neighbourhood of the main resonance $n = \frac{1}{2}$ is determined by the parameter $d = \delta(\epsilon)/(a_2 \epsilon E_0^{\frac{1}{2}})$. A full horizontal line denotes existence of the bifurcations, a full vertical line denotes where the type II or type III bifurcations branch off the normal mode (type I). Stability is indicated by ○, instability by ×. The figure is based on the analysis in §§7.1, 7.2.

7.2. *Stability*

The orbital stability of the periodic solutions which have been found, is studied with the aid of the integral (16*a*). In the case of the type II and type III local bifurcations the periodic solutions correspond with critical points of the vector field describing the flow in the $\tilde{A}$, $\tilde{X}$ phase plane. The integral (16*a*) is a Morse function on open sets in the $\tilde{A}$, $\tilde{X}$ phase plane and the stability is established by considering the index of the critical points. This is performed by expanding the integral in the neighbourhood of the critical points while only keeping the quadratic terms. A definite quadratic form corresponds with index zero and implies stability, an indefinite quadratic form implies instability. This procedure is equivalent to the use of the integral as a Lyapunov function. The Lyapunov stability in $\mathbb{R}^2$ ($\tilde{A}$, $\tilde{X}$ plane) is weakened to orbital stability when applied to the periodic solutions in the original phase-space in $\mathbb{R}^4$. The type I (normal mode) bifurcation corresponds with the boundary $\tilde{A} = (E_0/2)^{\frac{1}{2}}$ of the $\tilde{A}$, $\tilde{X}$ plane and has to be studied in another way.

Type II periodic solution, $\tilde{X} = 0$, $\tilde{A}$ and $\tilde{B}$ given by equation (19): expansion of the integral (16*a*) yields as a condition for stability

$$3\tilde{A} + E_0^{\frac{1}{2}} d > 0$$

in which $\tilde{A}$ is given by equation (19). This result means that the type II periodic solution is stable for $d > -\frac{1}{2}\sqrt{2}$.

Type III periodic solution, $\tilde{X} = \pi$, $\tilde{A}$ and $\tilde{B}$ given by equation (21): expansion of the integral (16*a*) yields as a condition for stability

$$3\tilde{A} - E_0^{\frac{1}{2}} d > 0$$

in which $\tilde{A}$ is given by equation (21). This condition means that the type III periodic solution is stable for $d < \frac{1}{2}\sqrt{2}$.

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Type I normal mode solution: equation (14*a*) can, with the aid of the energy integral (equation (15)), be written as
$$\frac{d\tilde{X}}{d\tau} = \frac{\delta(\epsilon)}{\epsilon} + a_2 \frac{6\tilde{A}^2 - E_0}{4\tilde{A}} \cos \tilde{X}.$$

We consider the cylindrical phase-space obtained from the $\tilde{A}$, $\tilde{X}$ plane by identifying $\tilde{X} = 0$ and $\tilde{X} = 2\pi$. A D_η neighbourhood of the normal mode $\tilde{A} = (\frac{1}{2}E_0)^{\frac{1}{2}}$ is given by $\frac{1}{2}E_0 - \eta < \tilde{A}^2 \leqslant \frac{1}{2}E_0$ with η a small positive parameter. We consider two cases:

If
$$0 < a < \left| \frac{\delta(\epsilon)}{\epsilon} + a_2 \frac{6\tilde{A}^2 - E_0}{4\tilde{A}} \cos \tilde{X} \right| < b < \infty,$$

where a and b are positive constants in D_η for η sufficiently small, $|\tilde{X}(\tau)|$ is a monotonically increasing function of τ without upper bound. It then follows from the integral (16*a*) and the implicit function theorem that A is a 2π-periodic function of $\tilde{X}$ in the neighbourhood of $\tilde{A} = (\frac{1}{2}E_0)^{\frac{1}{2}}$. This implies orbital stability of the normal mode. If
$$\frac{\delta(\epsilon)}{\epsilon} + a_2 \frac{6\tilde{A}^2 - E_0}{4\tilde{A}} \cos \tilde{X}$$

has a zero in D_η for each small positive η, we infer instability of the normal mode (we still have to check whether we have a turning point or not by calculating the second derivative).

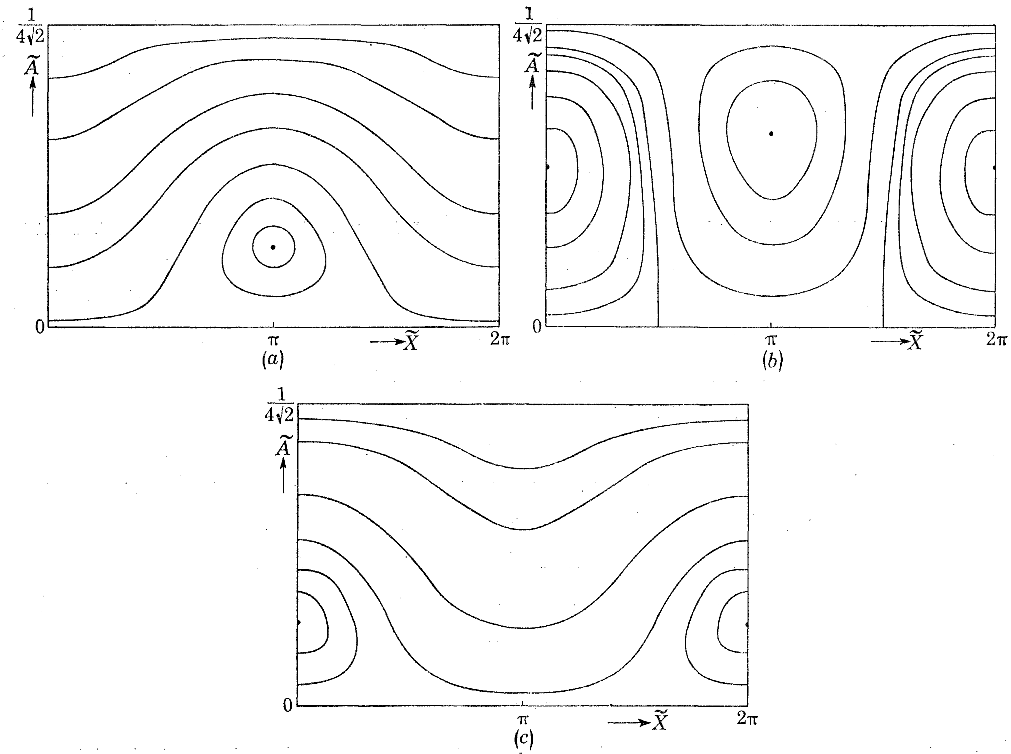

FIGURE 2. The $\tilde{A}$, $\tilde{X}$ phase plane near the main resonance $n = \frac{1}{2}$ derived from equation (16*a*) in three characteristic cases; energy $E_0 = \frac{1}{16}$. A dot denotes a critical point in the phase plane, corresponding with a periodic solution. The existence and stability of the periodic solutions (bifurcations) is discussed in §7.

Case *a*. $d = -1$, $\delta(\epsilon) = \epsilon$, $a_2 = -4$; there are two stable periodic solutions, i.e. the normal mode $\tilde{A} = 1/4\sqrt{2}$ and the solution at $\tilde{X} = \pi$.

Case *b*. $d = \frac{1}{9}$, $\delta(\epsilon) = \frac{1}{6}\epsilon$, $a_2 = 6$; there are two stable periodic solutions (at $\tilde{X} = 0, \pi$) and one unstable normal mode solution $\tilde{A} = 1/4\sqrt{2}$.

Case *c*. $d = +1$, $\delta(\epsilon) = \epsilon$, $a_2 = 4$; there are two stable periodic solutions, i.e. the normal mode $\tilde{A} = 1/4\sqrt{2}$ and the solution at $\tilde{X} = 0$.

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Putting $\tilde{A}^2 = E_0/2 - \bar{A}^2$ with $0 \leqslant \bar{A}^2 < \eta$ we obtain the equation

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} = \frac{\delta(\epsilon)}{\epsilon} + a_2 (\tfrac{1}{2}E_0)^{\frac{1}{2}} \cos \tilde{X} + O(\bar{A}).$$

So we find instability of the normal mode if $|d| < \frac{1}{2}\sqrt{2}$, stability if $|d| > \frac{1}{2}\sqrt{2}$.

The existence and stability characteristics of the bifurcations are summarized in figure 1. The phase flow in the $\tilde{A}$, $\tilde{X}$ plane is depicted for the three characteristic cases $d < -\frac{1}{2}\sqrt{2}$,

$$|d| < \tfrac{1}{2}\sqrt{2}, \quad d > \tfrac{1}{2}\sqrt{2}$$

in figure 2.

7.3 *The case* $a_2 = 0$

In this case the right-hand sides of equations (14) become zero and the amplitudes and phases can be approximated by their initial values on the time scale $1/\epsilon$. Clearly, significant changes in the dynamical system take place on a longer time scale. Condition (11) is satisfied and we introduce the modified Birkhoff transformation (12). Averaging of the resulting equations produces for the approximate amplitudes and phases

$$\left.\begin{aligned} &\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} = O(\epsilon^3), \quad \frac{\mathrm{d}\tilde{B}}{\mathrm{d}t} = O(\epsilon^3), \\ &\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t} = -\epsilon^2(\tfrac{5}{24}a_1^2 + \tfrac{3}{16}b_1)\,\tilde{A}^2 - \tfrac{1}{8}\epsilon^2 b_2 \tilde{B}^2 + \delta(\epsilon) + O(\epsilon^3), \\ &\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} = -\tfrac{1}{4}\epsilon^2 b_2 \tilde{A}^2 - \epsilon^2 \tfrac{3}{8} b_3 \tilde{B}^2 + O(\epsilon^3). \end{aligned}\right\} \tag{23}$$

Because of the increased order of approximation it is useful to admit only smaller orders of detuning: $\delta(\epsilon) = O(\epsilon^2)$. It follows from equations (23) and §5 that, if $a_2 = 0$, $A(t) - A(0) = O(\epsilon)$, $B(t) - B(0) = O(\epsilon)$ on the time scale $1/\epsilon^2$. The variation with time of $\tilde{\phi}(\epsilon^2 t)$ and $\tilde{\psi}(\epsilon^2 t)$ can be obtained by replacing $\tilde{A}$ and $\tilde{B}$ in equations (23) by $A(0)$ and $B(0)$ and integrating. The resulting $\tilde{x}(\epsilon^2 t)$ and $\tilde{z}(\epsilon^2 t)$ are approximations with error $O(\epsilon)$ on the time scale $1/\epsilon^2$. Some other conclusions are evident if $a_2 = 0$:

(1) Instead of the total energy of the system and the approximate integral I_3 we have two independent approximate integrals corresponding with the respective energies in each of the two degrees of freedom. This degeneration causes a drastic change of the topology of phase space.

(2) The two normal modes $\tilde{A} = 0$ ($z, \dot{z}$ degree of freedom) and $\tilde{B} = 0$ ($x, \dot{x}$ degree of freedom) do both exist. It is easy to verify that no small denominator problems exist in this case.

(3) The description of the phase-flow is not complete and needs approximations on a longer time scale than $1/\epsilon^2$. This subject falls within the scope of higher-order resonances and will be treated in a subsequent paper.

The consequences of the results of §§6 and 7 in the perspective of structural stability will be summarized in §12.

8. The resonance case $n = 1$

We introduce the time-like variable $\tau = \epsilon^2 t$ and the auxiliary angular variable X by

$$X = 2(\phi - \psi).$$

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8.1. *The amplitude–phase equations after transformation*

The introduction of the modified Birkhoff transformation (12) into equations (10) and averaging produces after lengthy calculations equations for the first asymptotic approximations of the amplitudes and phases:

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tau} = (\tfrac{1}{12}a_1 a_2 - \tfrac{1}{2}a_2^2 - \tfrac{1}{8}b_2)\,\tilde{A}\tilde{B}^2 \sin \tilde{X}, \tag{24a}$$

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau} = -(\tfrac{5}{12}a_1^2 + \tfrac{3}{8}b_1)\,\tilde{A}^2 - (\tfrac{1}{2}a_1 a_2 + \tfrac{1}{3}a_2^2 + \tfrac{1}{4}b_2)\,\tilde{B}^2 + (\tfrac{1}{12}a_1 a_2 - \tfrac{1}{2}a_2^2 - \tfrac{1}{8}b_2)\,\tilde{B}^2 \cos \tilde{X}, \tag{24b}$$

$$\frac{\mathrm{d}\tilde{B}}{\mathrm{d}\tau} = -(\tfrac{1}{12}a_1 a_2 - \tfrac{1}{2}a_2^2 - \tfrac{1}{8}b_2)\,\tilde{A}^2 B \sin \tilde{X}, \tag{24c}$$

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tau} = -(\tfrac{1}{2}a_1 a_2 + \tfrac{1}{3}a_2^2 + \tfrac{1}{4}b_2)\,\tilde{A}^2 - (\tfrac{5}{12}a_2^2 + \tfrac{3}{8}b_3)\,\tilde{B}^2 + \frac{1}{2}\frac{\delta(\epsilon)}{\epsilon^2} + (\tfrac{1}{12}a_1 a_2 - \tfrac{1}{2}a_2^2 - \tfrac{1}{8}b_2)\,\tilde{A}^2 \cos \tilde{X}. \tag{24d}$$

We choose $\delta(\epsilon) = O(\epsilon^2)$ as indicated in §5. The initial conditions are the same as those imposed on A, ϕ, B and ψ. For the approximate quantities we have estimates of the form

$$A(t) - \tilde{A}(\epsilon^2 t) = O(\epsilon)$$

on the time scale $1/\epsilon^2$, etc. The right-hand sides of equations (24) agree with results obtained earlier for the Hénon–Heiles problem (Verhulst 1977). These results in their turn were checked independently by applying the classical Birkhoff transformation to the vector field instead of the modified transformation (12).

The equations for $\tilde{\phi}$ and $\tilde{\psi}$ can be used to obtain the equation for the phase difference $\tilde{X}$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} = (-\tfrac{5}{6}a_1^2 + a_1 a_2 + \tfrac{2}{3}a_2^2 - \tfrac{3}{4}b_1 + \tfrac{1}{2}b_2)\,\tilde{A}^2 - (a_1 a_2 - \tfrac{1}{6}a_2^2 + \tfrac{1}{2}b_2 - \tfrac{3}{4}b_3)\,\tilde{B}^2 - \frac{\delta(\epsilon)}{\epsilon^2} + (\tfrac{1}{6}a_1 a_2 - a_2^2 - \tfrac{1}{4}b_2)\,(\tilde{B}^2 - \tilde{A}^2)\cos \tilde{X}. \tag{25}$$

8.2. *Integrals of motion*

Not unexpectedly the equations for $\tilde{A}$ and $\tilde{B}$ produce, after integration, the approximate energy integral. Multiplication of equation (24*a*) by A and equation (24*c*) by $\tilde{B}$, addition of the equations and integration yields

$$\tilde{A}^2(\tau) + \tilde{B}^2(\tau) = 2E_0, \tag{26}$$

where E_0 is a constant determined by the initial conditions. It is easy to show that

$$A^2(t) + B^2(t) - 2E_0 = O(\epsilon) \quad \text{on the time scale } 1/\epsilon^2.$$

As in §6.2 we have a much stronger result; the boundedness of the solutions on the energy manifold (ϵ is small) enables us to conclude that this estimate holds uniformly for $t > 0$.

A second integral of system (24) can be obtained as follows. We use the energy integral (26) to eliminate $\tilde{B}$ from equations (24*a*) and (25) to obtain

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tau} = (\tfrac{1}{12}a_1 a_2 - \tfrac{1}{2}a_2^2 - \tfrac{1}{8}b_2)\,\tilde{A}(2E_0 - \tilde{A}^2)\sin \tilde{X}, \tag{24aa}$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} = -(a_1 a_2 - \tfrac{1}{6}a_2^2 + \tfrac{1}{2}b_2 - \tfrac{3}{4}b_3)\,2E_0 - \frac{\delta(\epsilon)}{\epsilon^2} + (-\tfrac{5}{6}a_1^2 + 2a_1 a_2 + \tfrac{1}{2}a_2^2 - \tfrac{3}{4}b_1 + b_2 - \tfrac{3}{4}b_3)\,\tilde{A}^2 + (\tfrac{1}{6}a_1 a_2 - a_2^2 - \tfrac{1}{4}b_2)\,2(E_0 - \tilde{A}^2)\cos \tilde{X}. \tag{25a}$$

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The equations for $\tilde{A}$ and $\tilde{X}$ admit an integral of motion independent of the energy if

$$b_2 \neq \tfrac{2}{3}a_1 a_2 - 4a_2^2. \tag{27}$$

If condition (27) is satisfied we have

$$\tilde{A}^2(\tau)\,[(\tilde{A}^2(\tau) - 2E_0)\cos\tilde{X}(\tau) + \alpha\tilde{A}^2(\tau) + \beta] = I_3 \tag{28a}$$

in which
$$\alpha = \frac{\frac{5}{6}a_1^2 - 2a_1 a_2 - \frac{1}{2}a_2^2 + \frac{3}{4}b_1 - b_2 + \frac{3}{4}b_3}{\frac{1}{3}a_1 a_2 - 2a_2^2 - \frac{1}{2}b_2},$$

$$\beta = \frac{(a_1 a_2 - \frac{1}{6}a_2^2 + \frac{1}{2}b_2 - \frac{3}{4}b_3)\,2E_0 + \delta(\epsilon)/\epsilon^2}{\frac{1}{6}a_1 a_2 - a_2^2 - \frac{1}{4}b_2}.$$

Another form of the integral (28*a*) will prove to be useful. With the energy integral (26) we may write the second integral as

$$\tilde{A}(\tau)\,[\tilde{B}^2(\tau)\,(-\alpha - \cos\tilde{X}(\tau)) + \gamma] = I_3, \tag{28b}$$

in which
$$\gamma = \frac{\delta(\epsilon)/\epsilon^2 + (\frac{5}{6}a_1^2 - \frac{5}{6}a_2^2 + \frac{3}{4}b_1 - \frac{3}{4}b_3)\,E_0}{\frac{1}{6}a_1 a_2 - a_2^2 - \frac{1}{4}b_2}.$$

The constant I_3 is determined by the initial conditions.

In the sequel the integral (28) will play an important part; at this stage we make the following remarks.

(1) It follows from the theory of asymptotic approximations referred to in §5 that we have the following estimate

$$A^2(t)\,[B^2(t)\,(-\alpha - \cos X(t) + \gamma)] - I_3 = O(\epsilon) \quad \text{on the time scale } 1/\epsilon^2.$$

A similar estimate can be deduced for expression (28*a*). It would be useful if the asymptotic validity of the integral could be extended over the whole time axis. It is not clear how to perform this by the methods of asymptotic analysis alone (for a discussion of extension methods in asymptotic analysis see Verhulst 1976). Even in the case of the approximate energy integral (26), where the extension of the time scale is nearly trivial, we had to use a geometric argument to prove the uniform validity. Unfortunately Arnold's (1963) theorem on the uniform validity of first integrals in Hamiltonian systems does not apply to this case.

(2) The relation between the second approximate integral of motion and the angular momentum of the nearly periodic motion around the origin of phase-space is clear only in a special case. If $\alpha = -1$, $\gamma = 0$, which happens for instance in the Hénon–Heiles problem, the integral (28*b*) becomes in cartesian variables after some minor rearrangements

$$I_3 = 2(x\dot{z} - z\dot{x})^2.$$

(3) The integral (28) does not exist if condition (27) is not satisfied, i.e. if

$$b_2 = \tfrac{2}{3}a_1 a_2 - 4a_2^2.$$

The equations (24) for the approximate amplitudes and phases become in this case

$$\frac{d\tilde{A}}{d\tau} = \frac{d\tilde{B}}{d\tau} = 0,$$

$$\frac{d\tilde{\phi}}{d\tau} = -(\tfrac{5}{12}a_1^2 + \tfrac{3}{8}b_1)\,\tilde{A}^2 - \tfrac{2}{3}(a_1 a_2 - a_2^2)\,\tilde{B}^2,$$

$$\frac{d\tilde{\psi}}{d\tau} = -\tfrac{2}{3}(a_1 a_2 - a_2^2)\,\tilde{A}^2 - (\tfrac{5}{12}a_2^2 + \tfrac{3}{8}b_3)\,\tilde{B}^2 + \frac{1}{2}\frac{\delta(\epsilon)}{\epsilon^2}.$$

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If the parameters are chosen such that the right-hand sides of the equations for $\tilde{\phi}$ and $\tilde{\psi}$ vanish, we may expect resonance on the time-scale $1/\epsilon^3$. To study this type of resonance one must include the third-order terms in ϵ in the equations of motion (the fifth-order terms in the potential U_3). If these right-hand sides do not vanish, one has no main resonance for $n = 1$ and the problem should be treated as a higher-order resonance. Such resonances will be considered in a subsequent paper.

9. Bifurcations in the resonance case $n = 1$

In this section we shall look for the periodic solutions which branch off in the case of perturbations of the two-dimensional harmonic oscillator given by equation (8). Again, we shall characterize these bifurcations for fixed but arbitrary values of the energy (arbitrary within the scope of asymptotic analysis in the neighbourhood of the origin of phase-space). In equations (24) six parameters occur, so we expect five-parameter families of bifurcations. To study these we shall give the general existence and stability criteria (the concept of stability is used here in the sense of orbital stability). We shall apply these to a one-parameter family of bifurcations studied by Braun (1973) and we shall consider two- and higher-order parameter families. A somewhat special role is played by the detuning parameter $\delta(\epsilon)$ and we shall give this parameter special attention.

9.1. *Existence of local and global bifurcations*

Type I: two normal mode solutions.

Putting $\tilde{A}(\tau) \equiv 0$ or $\tilde{B}(\tau) \equiv 0$ in equations (24) produces two periodic solutions. It is easy to verify (see §7) that there are no small denominator problems. From the equations of motion (8) it is concluded that the approximate solution $\tilde{A}(\tau) \equiv 0$ does not correspond with an exact normal mode in the $x = \dot{x} = 0$ plane but with a nearly normal mode periodic solution. The approximate solution $\tilde{B}(\tau) \equiv 0$, however, corresponds with an exact normal mode in the $z = \dot{z} = 0$ plane; this is a consequence of the discrete symmetry of the potential U_3 in z. We shall return in more detail to this exact periodic solution in §9.2 and in a subsequent paper.

Type II: two periodic solutions if $\tilde{X} = 0, 2\pi$ and condition (27) is satisfied.

The right-hand side of equation (25*a*) and the condition $0 < \tilde{A}^2 < 2E_0$ produce the condition of existence

$$0 < \frac{10a_2(a_1+a_2)+9(b_2-b_3)+6\delta(\epsilon)/(E_0\epsilon^2)}{10(a_1+a_2)(3a_2-a_1)-9(b_1-2b_2+b_3)} < 1. \tag{29}$$

It is clear that the detuning parameter $\delta(\epsilon)$ plays a role in the existence of type II periodic solutions which becomes increasingly important for smaller values of the energy. We shall return to condition (27) while studying specific parameter families.

Type III: two periodic solutions if $\tilde{X} = \pi, 3\pi$ and condition (27) is satisfied.

The right-hand side of equation (25*a*) and the condition $0 < \tilde{A}^2 < 2E_0$ produce the condition of existence

$$0 < \frac{14a_2(a_1-a_2)+3(b_2-3b_3)+6\delta(\epsilon)/(E_0\epsilon^2)}{2(a_1-a_2)(9a_2-5a_1)-3(3b_1-2b_2+3b_3)} < 1. \tag{30}$$

Again the influence of the detuning parameter $\delta(\epsilon)$ becomes more important for smaller values of the energy.

The three types of periodic solutions described here are all local bifurcations with respect to the energy, i.e. for each arbitrary but fixed value of the energy, we find 2, 4 or 6 periodic solutions. That we always find at least two bifurcations is in accordance with the existence theorem by

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Weinstein (1973) for Hamiltonian systems. Apart from these local solutions, global bifurcations with respect to the energy may arise in the following way. We look for solutions with $\sin \tilde{X} = 0$ for all time while at the same time equation (25 *a*) is satisfied for all permissible values of the amplitudes. This induces the following cases:

$\tilde{X} = 0, 2\pi$,

while simultaneously
$$\left.\begin{aligned} 10a_2(a_1+a_2)+9(b_2-b_3)+\frac{6\delta(\epsilon)}{E_0\,\epsilon^2} &= 0,\\ 10(a_1+a_2)\,(3a_2-a_1)-9(b_1-2b_2+b_3) &= 0; \end{aligned}\right\} \tag{31}$$

$\tilde{X} = \pi, 3\pi$,

while simultaneously
$$\left.\begin{aligned} 14a_2(a_1-a_2)+3(b_2-3b_3)+\frac{6\delta(\epsilon)}{E_0\,\epsilon^2} &= 0,\\ 2(a_1-a_2)\,(9a_2-5a_1)-3(3b_1-2b_2+3b_3) &= 0. \end{aligned}\right\} \tag{32}$$

These global bifurcations can be studied by looking at the corresponding form of the integral of motion (28 *a*) or (28 *b*):

(1) $\tilde{X} = 0,\ 2\pi$ while equations (31) hold. We find, surprisingly, that the coefficients do not depend any more on the parameters: $\alpha = -1$, $\gamma = 0$. The integral (28 *b*) becomes

$$I_3 = \tilde{A}^2(\tau)\,\tilde{B}^2(\tau)\,[1-\cos \tilde{X}(\tau)]. \tag{33}$$

In cartesian variables the integral becomes after some minor rearrangements $I_3 = 2(x\dot{z}-\dot{x}z)^2$. So in the case of these global bifurcations the second integral reduces to the angular momentum integral of the linear system. It is easy to deduce from the integral that at the same time two type III local bifurcations exist (we return to this in § 9.3).

(2) $\tilde{X} = \pi, 3\pi$ while equations (32) hold. Again we find that the coefficients in the integral of motion (28 *a*, *b*) do not depend any more on the parameters. We find: $\alpha = 1$, $\gamma = 0$. The integral (28 *b*) becomes

$$I_3 = -\tilde{A}^2(\tau)\,\tilde{B}^2(\tau)\,[1+\cos \tilde{X}(\tau)]. \tag{34}$$

In cartesian variables the integral becomes after some minor rearrangements $I_3 = -2(xz+\dot{x}\dot{z})^2$. It is easy to deduce from the integral that at the same time two type II local bifurcations exist (we return to this in § 9.3).

9.2. *Exact solutions*

Although equations (5) (or (8)) are nonlinear a few exact solutions can be found. These solutions are of particular importance as they can be used to demonstrate the non-uniform dependence of the existence of periodic solutions on the energy. It turns out that the condition of discrete symmetry of the potential U_3 in z preserves the normal mode in the $(x, \dot{x})$ degree of freedom under nonlinear perturbations.

Putting $u = z^2$, equations (5) become

$$\ddot{x} = -\frac{\partial U_3}{\partial x}(x, z^2), \quad \ddot{z} = -\frac{\partial U_3}{\partial u}(x, u)\bigg|_{u=z^2} 2z.$$

Substitution of $z = \dot{z} = 0$ yields
$$\ddot{x} = -\partial U_3(x, 0)/\partial x. \tag{35}$$

The solutions of equation (35) are periodic in the vicinity of the origin of phase-space and correspond with the type I periodic solutions in the case $\tilde{B}(\tau) \equiv 0$. We remark that these solutions exist for all values of ω.

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A more special case is the following. Retain only the quadratic and cubic terms in the potential (7) (in equation (7) $b_1 = b_2 = b_3 = \ldots = 0$) and choose $\omega = 1$. The equations of motion become

$$\ddot{x}+x = a_1 x^2+a_2 z^2,$$
$$\ddot{z}+z = 2a_2 xz.$$

Inspired by the results on local bifurcations in §9.1 we look for solutions of the form

$$\phi(t) = \psi(t) \quad \text{and} \quad \phi(t) = \psi(t)+\pi.$$

Substitution in the corresponding phase-amplitude equations (10) with $n = 1$, $\delta(\epsilon) = 0$, $\epsilon = 1$, $b_1 = b_2 = b_3 = \ldots = 0$ produces for the amplitudes the relation

$$(2a_2-a_1)\, A^2(t) = a_2 B^2(t)$$

and so

$$(2a_2-a_1)\, x^2(t) = a_2 z^2(t). \tag{36}$$

Equation (36) corresponds with solutions of the equations of motion if

$$a_2(2a_2-a_1) > 0. \tag{37}$$

If $2a_2-a_1 = 0$ these solutions reduce to the normal mode solutions represented by equation (35); if $a_2 = 0$ the equations of motion are uncoupled. From equation (36) it is clear that if $x(t)$ is periodic or unbounded, $z(t)$ is periodic or unbounded and vice versa. Conditions of periodicity are obtained easily by considering the Hamiltonian (6) after elimination of z

$$h = \frac{3a_2-a_1}{2a_2}(x^2+\dot{x}^2-\tfrac{4}{3}a_2 x^3).$$

For each value of the energy this equation produces a cubic curve in the x, $\dot{x}$ plane. This curve consists of an open branch and a closed curve, corresponding with cycles (periodic solutions) around the origin if

$$0 < h < \frac{3a_2-a_1}{24a_2^3}.$$

9.3. *Stability*

As in §7.2 we shall use the first integrals which we found to study the orbital stability of the periodic solutions. Here, the type II and type III local bifurcations correspond with the critical points of the integral (28*a*). We use Morse theory to establish their stability. The type I (normal modes) bifurcations correspond with the boundaries $\tilde{A} = 0$, $\tilde{A} = (2E_0)^{\frac{1}{2}}$ in the $\tilde{A}$, $\tilde{X}$ phase plane and have to be studied in a different way.

In the case of the global bifurcations the critical points are degenerate, but as the integral simplifies rather drastically (cf. equations (33) and (34)) it is easy to see what happens. We find after some calculations:

Type II periodic solutions, $\tilde{X} = 0, 2\pi$: positive quadratic form, i.e. stability if,

$$(a_1 a_2-6a_2^2-\tfrac{3}{2}b_2)\,[\tfrac{5}{3}(a_1+a_2)\,(a_1-3a_2)+\tfrac{3}{2}(b_1-2b_2+b_3)] > 0. \tag{38}$$

The solutions are unstable if this expression is negative.

Type III periodic solutions, $\tilde{X} = \pi, 3\pi$: positive quadratic form, i.e. stability, if

$$(a_1 a_2-6a_2^2-\tfrac{3}{2}b_2)\,[2(a_1-a_2)\,(5a_1-9a_2)+3(3b_1-2b_2+3b_3)] < 0. \tag{39}$$

The solutions are unstable if this expression is positive.

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Remark: the detuning parameter $\delta(\epsilon)$ plays a part in determining the existence of the type II and type III periodic solutions but *not* in the stability of these solutions.

Type I (normal mode) solutions. To study the stability of these solutions we use the same method as developed in §7.2. This method involves the equation (25*a*) for $\tilde{X}$ and the integral (28*a*) together with the implicit function theorem. The $\tilde{B} = 0$, $\tilde{A}^2 = 2E_0$ normal mode is studied by considering equation (25*a*) in a D_η neighbourhood of $\tilde{A}^2 = 2E_0$ by putting $\tilde{A}^2 = 2E_0 - \bar{A}^2$, $0 \leqslant \bar{A}^2 < \eta$. Equation (25*a*) becomes

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} = (a_1 a_2 + \tfrac{2}{3}a_2^2 - \tfrac{5}{6}a_1^2 - \tfrac{3}{4}b_1 + \tfrac{1}{2}b_2)\, 2E_0 - \frac{\delta(\epsilon)}{\epsilon^2} - 2E_0(\tfrac{1}{6}a_1 a_2 - a_2^2 - \tfrac{1}{4}b_2)\cos\tilde{X} + O(\bar{A}^2).$$

So we have stability of the normal mode $\tilde{B} = 0$ if

$$\left|\frac{a_1 a_2 + \frac{2}{3}a_2^2 - \frac{5}{6}a_1^2 - \frac{3}{4}b_1 + \frac{1}{2}b_2 - \delta(\epsilon)/(2E_0\epsilon^2)}{\frac{1}{6}a_1 a_2 - a_2^2 - \frac{1}{4}b_2}\right| > 1. \tag{40}$$

We have instability if this condition is not satisfied. The $\tilde{A} = 0$, $\tilde{B}^2 = 2E_0$ normal mode is studied by considering equation (25*a*) in a D_η neighbourhood of $\tilde{A}^2 = 0$ by putting $0 \leqslant \tilde{A}^2 < \eta$. Equation (25*a*) becomes

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} = -(a_1 a_2 - \tfrac{1}{6}a_2^2 + \tfrac{1}{2}b_2 - \tfrac{3}{4}b_3)\, 2E_0 - \delta(\epsilon)/(\epsilon^2) + 2E_0(\tfrac{1}{6}a_1 a_2 - a_2^2 - \tfrac{1}{4}b_2)\cos\tilde{X} + O(\tilde{A}^2).$$

So we have stability of the normal mode $\tilde{A} = 0$ if

$$\left|\frac{a_1 a_2 - \frac{1}{6}a_2^2 + \frac{1}{2}b_2 - \frac{3}{4}b_3 + \delta(\epsilon)/(2E_0\epsilon^2)}{\frac{1}{6}a_1 a_2 - a_2^2 - \frac{1}{4}b_2}\right| > 1. \tag{41}$$

We have instability if this condition is not satisfied.

Some insight in the phase-flow in the $\tilde{A}$, $\tilde{X}$ plane can be gained by considering the separatrix with the initial conditions such that $I_3 = 0$. Equation (28*a*) produces in this case two orbits, i.e. $\tilde{A} = 0$ (a normal mode) and one given by

$$(\tilde{A}^2 - 2E_0)\cos\tilde{X} + \alpha\tilde{A}^2 + \beta = 0. \tag{42}$$

Whether the orbit corresponding with equation (42) exists or not depends on α and β. If this orbit intersects with the normal mode $\tilde{A} = 0$ this normal mode cannot be stable. At the intersection we have

$$-2E_0\cos\tilde{X} + \beta = 0.$$

So we have stability of the normal mode $\tilde{A} = 0$ if $|\beta/2E_0| > 1$ and instability if this condition is not satisfied. This result is equivalent with the condition represented by inequality (41).

The global bifurcations $\tilde{X} = 0, 2\pi$ (equations (20) hold) and $\tilde{X} = \pi, 3\pi$ (equations (21) hold) are unstable. This result is made explicit by plotting the $\tilde{A}^2$, $\tilde{X}$ phase plane portrait in figure 3 using the integrals (33) and (34) after eliminating $\tilde{B}^2(\tau)$. At the same time it is clear that $\tilde{X} = 0, 2\pi$ global bifurcations are accompanied by stable type III local bifurcations and unstable type I (normal mode) solutions. The $\tilde{X} = \pi, 3\pi$ global bifurcations are accompanied by stable type II local bifurcations and unstable type I solutions.

Both the existence and the stability considerations up till now have been based on the normal forms (24) of the equations of motion and the corresponding integrals. In obtaining these results the $O(\epsilon^3)$ terms in the equations (the 5th- and higher-order terms in the Hamiltonian and the potential) have been omitted and we wish to show that our results are essentially unchanged by the

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addition of these higher-order perturbations. To accomplish this we have to reduce the system of differential equations to a Poincaré mapping. This can be done by introducing action-angle variables or using iso-energetic reduction (Siegel & Moser 1971). In both cases we reduce to a one-degree-of-freedom Hamiltonian which is non-autonomous. The type II and type III bifurcations are found as critical points of the corresponding vector field. This way of studying the stability of these solutions has been explored by Braun (1973) for the case $b_1 = b_2 = b_3 = \delta(\epsilon) = 0$. He employs Moser's twist theorem to prove the existence of infinitely many invariant tori with non-zero measure on each energy surface which implies the stability. Braun's proof carries over without serious complications to our more general case. However, the critical points of the mapping have to be non-degenerate which leaves out the global bifurcations which we found.

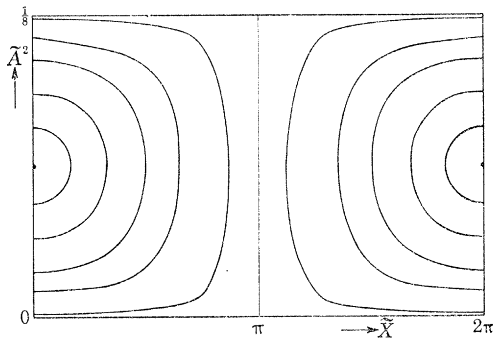

FIGURE 3. The $\tilde{A}^2$, $\tilde{X}$ phase plane near the main resonance $n = 1$ with global bifurcations at $\tilde{X} = \pi, 3\pi$. The orbital curves are obtained from equations (34) and (26) with $E_0 = \frac{1}{16}$. Critical points corresponding with stable periodic solutions are found at $\tilde{A}^2 = E_0$, $\tilde{X} = 0, 2\pi$. Each point of the line $\tilde{X} = \pi$ (or 3π) corresponds with a periodic solution. The analysis of existence and stability is given in §§9.1 and 9.3. The phase-flow near the $\tilde{X} = 0, 2\pi$ global bifurcations is obtained by translation of the $\tilde{X}$-axis by a factor π.

9.4. *Braun's one-parameter family of bifurcations; detuning*

Braun (1973) studied the phaseflow, corresponding with equations (5), (7), where $\omega = 1$, $b_1 = b_2 = b_3 = \ldots = 0$. We assume $a_2 \neq 0$ to avoid uncoupling of the equations. We shall show that Braun's results are contained in ours, moreover it will become clear that this discussion of the normal modes and the global bifurcations is not complete. The results are obtained by studying a one-parameter family of bifurcations; we extend these results to a two-parameter family by admitting detuning. The bifurcation parameter introduced by Braun is $\lambda = a_1/3a_2$. The parameters α and β (§8.2) become

$$\alpha = \frac{15\lambda^2 - 12\lambda - 1}{2(\lambda - 2)}, \quad \beta = \frac{18\lambda - 1}{3(\lambda - 2)} 2E_0.$$

Condition (27) is violated if $\lambda = 2$. As will become clear in a subsequent paper, we have in this case a higher-order resonance of an uninteresting type: the only bifurcations are the two normal mode periodic solutions. We assume here $\lambda \neq 2$. The analysis of existence and stability of local bifurcations given in §§9.1 and 9.3 is summarized in figure 4.

Remarks

(1) The global bifurcation at $\lambda = -\frac{1}{3}$ occurs as a degeneration of type II local bifurcations in the transition from unstable to stable solutions. At $\lambda = \frac{1}{3}$ the global bifurcation is prompted as an isolated event. The $\lambda = -\frac{1}{3}$ global bifurcation occurs in Braun's list, the $\lambda = \frac{1}{3}$ bifurcation does not.

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(2) As has been pointed out in § 9.1 we find at least two bifurcations for each value of λ (in accordance with the Weinstein (1973) theorem). In Braun's list the $\tilde{B} = 0$ normal mode is missing for $\lambda > \frac{2}{3}$. Using a different method, Kummer (1976) reached the same conclusion. It should be remarked here that this normal mode can even be recognized as an exact solution. Equation (35) becomes in this case $\ddot{x} + x = a_1 x^2$, producing periodic solutions if $0 < h < 1/(6a_1^2)$.

(3) The type II bifurcations have been given as exact periodic solutions in § 9.2. The condition of existence (37) again leads to the condition $\lambda < \frac{2}{3}$.

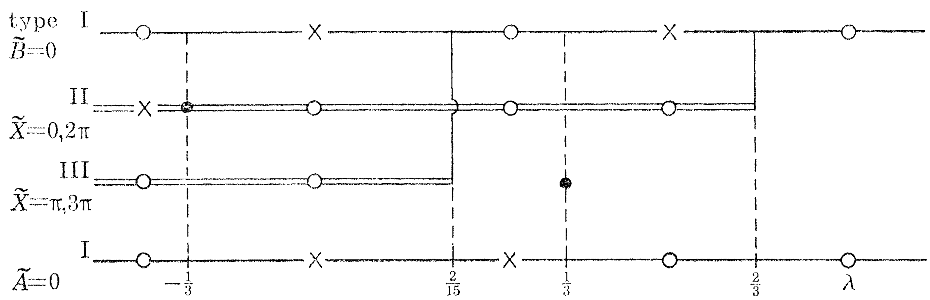

FIGURE 4. The existence of bifurcations (periodic solutions) for each value of the (small) energy at the main resonance $n = 1$ for Braun's one-parameter family; $\lambda = a_1/(3a_2)$, $a_2 \neq 0$, $\delta(\epsilon) = b_1 = b_2 = b_3 = 0$ in the potential (7). A full horizontal line denotes existence of the bifurcations, a full vertical line denotes where the type II or type III bifurcations branch off the normal mode $B = 0$. Stability is indicated by ○, instability by ×. A large dot at $\lambda = -\frac{1}{3}, \frac{1}{3}$ denotes the existence of a global bifurcation. The figure is based on the analysis in § 9.4.

(4) Some insight can be gained by looking at the normal modes as permanent phenomena and at the type II and III bifurcations as solutions branching off the normal modes. The following picture arises: For $\lambda > \frac{2}{3}$ only 2 normal modes exist; at $\lambda = \frac{2}{3}$ the type II bifurcation branches off the $\tilde{B} = 0$ normal mode (equation (36) or (29)); for $\lambda \to -\infty$ the type II bifurcation tends towards the $\tilde{A} = 0$ normal mode. A similar behaviour is found for the type III bifurcation from equation (30); for $\lambda = \frac{2}{15}$ the bifurcation branches off the $\tilde{B} = 0$ normal mode and for $\lambda \to -\infty$ it tends towards the $\tilde{A} = 0$ normal mode.

(5) In the a_1, a_2 parameter space the topology of the phase-flow induced by the corresponding Hamiltonian changes drastically on crossing the straight lines $a_2 = 0$, $a_1 + a_2 = 0$, $a_1 - \frac{2}{5}a_2 = 0$, $a_1 - a_2 = 0$, $a_1 - 2a_2 = 0$.

We now look at the effect of detuning on Braun's family of bifurcations. Consider the three parameters $a_1, a_2, \delta(\epsilon)$. We assume again $a_2 \neq 0$ to avoid uncoupling of the equations; this enables us to consider the effect of detuning in a two-parameter family of bifurcations. We put

$$d = \delta(\epsilon)/(a_2^2 E_0 \epsilon^2);$$

the existence condition (29) for type II bifurcations becomes

$$0 < \frac{3\lambda + 1 + \frac{3}{5}d}{3(3\lambda+1)(1-\lambda)} < 1.$$

The non-degeneracy condition (27) implies $\lambda \neq 2$. The $\tilde{X} = 0$, 2π global bifurcations are obtained from equations (31) and are found at $\lambda = -\frac{1}{3}$, $d = 0$ and $\lambda = 1$, $d = -\frac{20}{3}$. The domain of existence of the type II bifurcations is sketched in figure 5 (a). The stability is concluded from equation (38).

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The dependence of the parameter d on both the detuning parameter $\delta(\epsilon)$ and the energy E_0 causes another phenomenon. For example choose $\lambda = 0$; from figure 5(*a*) we know that type II bifurcations exist if $-\frac{5}{3} < d < \frac{10}{3}$. This inequality implies that if $\delta(\epsilon) > 0$ we have

$$E_0 > 3\delta(\epsilon)/(10a_2^2\epsilon^2); \quad \text{if } \delta(\epsilon) < 0 \quad \text{we have} \quad E_0 > -3\delta(\epsilon)/(5a_2^2\epsilon^2).$$

This means that in the detuned case, starting with energy E_0 very small, no type II bifurcations exist. Increasing the energy produces these bifurcations at a specific value of the energy.

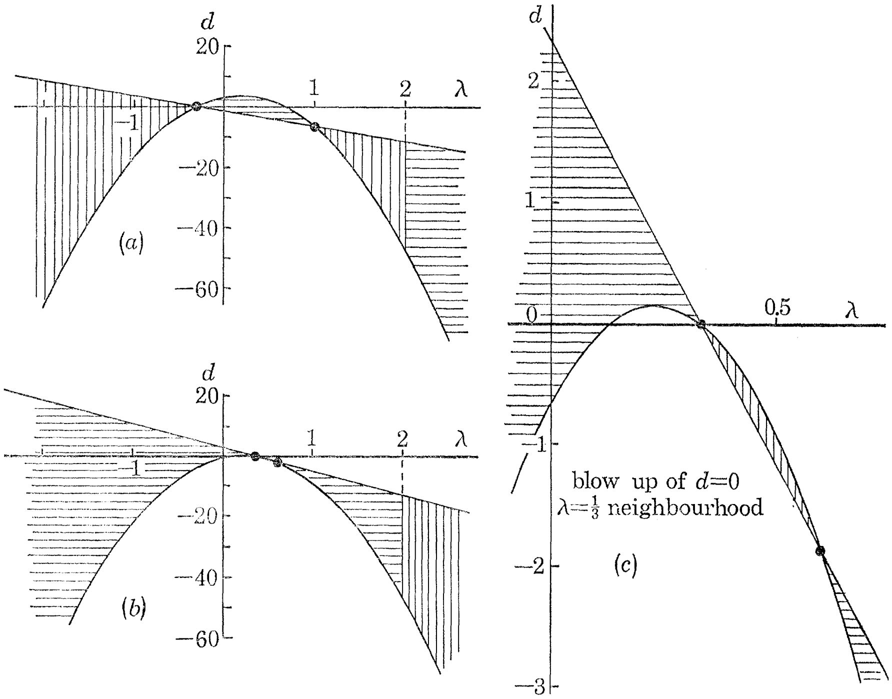

FIGURE 5. The existence domain of bifurcations in a two-parameter family obtained by detuning Braun's 1-parameter family; $\lambda = a_1/3a_2$, $d = \delta(\epsilon)/(a_2^2 E_0 \epsilon^2)$. (*a*) refers to type II bifurcations; (*b*) to type III bifurcations. Each point in a vertically shaded area denotes the existence of two unstable bifurcations, each point in a horizontally shaded area denotes two stable ones. There are four global bifurcations: type II at $\lambda = -\frac{1}{3}$, $d = 0$ and $\lambda = 1$, $d = -\frac{20}{3}$; type III at $\lambda = \frac{1}{3}$, $d = 0$ and $\lambda = \frac{3}{5}$, $d = -\frac{28}{15}$. Putting $d = 0$, one recovers the type II and type III existence lines in figure 4. The analysis is based upon the results of §9.4. (*c*) the blown-up neighbourhood of the two global bifurcations in figure 5(*b*).

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Similar results are obtained in studying the type III bifurcations. The condition of existence (30) becomes

$$0 < \frac{7(3\lambda-1)+3d}{3(3\lambda-1)(3-5\lambda)} < 1.$$

The non-degeneracy condition (27) implies again $\lambda \neq 2$. The $\tilde{X} = \pi$, 3π global bifurcations are obtained from equations (32) and are found at $\lambda = \frac{1}{3}$, $d = 0$ and $\lambda = \frac{3}{5}$, $d = -\frac{28}{15}$. The domain of existence of the type III bifurcations is sketched in figure 5(*b*). The stability is concluded from equation (39).

9.5. *Parameter families of bifurcations*

To understand the role of each of the parameters a_1, a_2, b_1, b_2, b_3 and $\delta(\epsilon)$ one has to study all the possible parameter families. We shall indicate families by degenerate if condition (27) is violated, i.e. if

$$b_2 = \tfrac{2}{3}a_2(a_1 - 6a_2). \tag{43}$$

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We remark that uncoupling of the equations of motion (8) to order ϵ^2 by taking $a_2 = b_2 = 0$ constitutes a special case of degeneracy according to equation (43). In our terminology the a_1, a_2 parameters will generate two degenerate cases $a_2 = 0$, $a_1 = 6a_2$ ($\lambda = 2$) and a one-parameter family of bifurcations (Braun's parameter family). In the same way the a_1, a_2, $\delta(\epsilon)$ parameters will generate two degenerate cases $a_2 = 0$, $a_1 = 6a_2$ and a two-parameter family of bifurcations (figure 5). The a_2, b_2, $\delta(\epsilon)$ parameters generate the degenerate case $b_2 + 4a_2^2 = 0$, for $a_2 \neq 0$ a two-parameter family of bifurcations, and for $a_2 = 0$ a one-parameter family, etc. In the following table we indicate the number of possible parameter families for the six parameters; because of the special role of detuning we indicate in each case the number of parameter families involving $\delta(\epsilon)$. To give an impression of the part played by degeneracy we give the number of uncoupled degenerate families ($a_2 = b_2 = 0$).

	number in parameter families				
	1	2	3	4	5
number of families	15	20	15	6	1
detuned families	5	10	10	5	1
uncoupled degenerate families	6	4	1	0	0

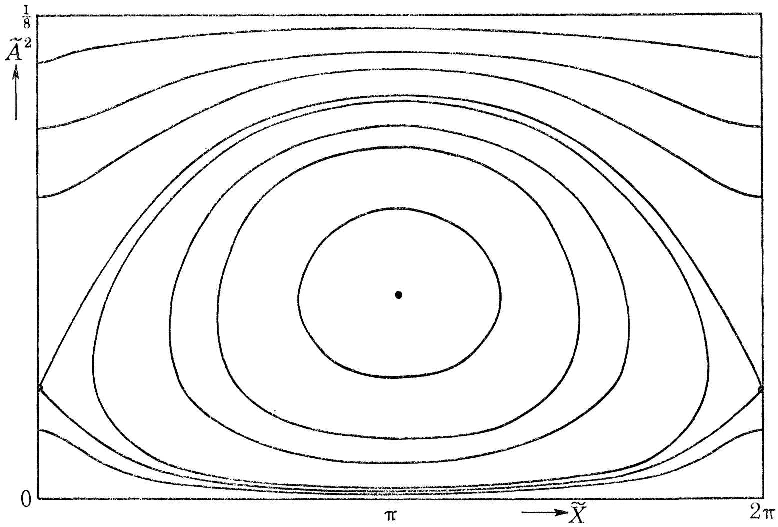

FIGURE 6. The $\tilde{A}^2$, $\tilde{X}$ phase plane at the main resonance $n = 1$ ($\delta(\epsilon) = 0$) in the case of Braun's one-parameter family where $\lambda = -\frac{1}{2}$. The orbital curves are obtained from equation (28a) with $E_0 = \frac{1}{16}$, $\alpha = -\frac{7}{4}$, $\beta = \frac{1}{6}$. The normal modes $\tilde{A}^2 = \frac{1}{8}$ and $\tilde{A}^2 = 0$ are stable periodic solutions. The critical points at $\tilde{X} = \pi, 3\pi$ (the last one is omitted) correspond with two stable periodic solutions, the critical points at $\tilde{X} = 0, 2\pi$ correspond with two unstable periodic solutions. The analysis of existence and stability is given in §§9.1, 9.3 and 9.4.

The one-parameter families of type II and type III bifurcations will be empty (degenerate families), or will consist of semi-infinite sets in $\mathbb{R}$ (e.g. Braun's parameter family, figure 4) or will consist of compact sets in $\mathbb{R}$ (fix, for instance, $a_1 = \lambda = 0$ in figure 5). It is seen from equations (29) and (30) that the 16 non-degenerate two-parameter families in $\mathbb{R}^2$ are bounded by straight lines and conic sections. We have seen an example of such a family by detuning Braun's parameter family (figure 5). Another example is the family produced by the parameters a_1, a_2 and b_2. If $a_1 a_2 = 0$ the two-parameter families are bounded by straight lines. The global bifurcations $\tilde{X} = 0, 2\pi$ and $\tilde{X} = \pi, 3\pi$ are found by solving two quadratic equations in each case (equations (31–32)). This means that among the one-and two-parameter families the global bifurcations are

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represented by 0, 1 or 2 isolated points in $\mathbb{R}$ or $\mathbb{R}^2$ respectively. On adding parameters, the dimension of the global bifurcations set grows. For instance, it can be seen that the three-parameter family induced by a_1, a_2, b_1 and $\delta(\epsilon)$ in $\mathbb{R}^3$ contains a global bifurcation set consisting of the intersection of a plane and a parabolic cylinder in $\mathbb{R}^3$. It is finally concluded that the three-, four- and five-parameter families in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^5$ respectively, consist of semi-infinite and compact sets which are connected by global bifurcation sets which have the dimension 1, 2 and 3 respectively.

The consequences of these considerations in the perspective of structural stability will be summarized in §12. In Braun's (1973) paper the phase-flow is illustrated for different values of λ; for reasons of comparison and to make the part played by the amplitudes more explicit we present the phase-flow in the $\tilde{A}^2$, $\tilde{X}$ phase plane. Our figure 3 for the global bifurcations corresponds with Braun's figures 4 and 7; in our figure 6 we present the case $\lambda = -\frac{1}{2}$ corresponding with Braun's figure 3.

10. The behaviour of the orbits with time in the case $n = 1$

The results of the preceding sections can be used to determine the behaviour of the orbits with time to an $O(\epsilon)$ approximation on the time-scale $1/\epsilon^2$. We summarize the results for the periodic orbits.

Type I (normal modes):

$$\left.\begin{aligned} \tilde{B}(\epsilon^2 t) \equiv 0, \quad \tilde{A}(\epsilon^2 t) &= (2E_0)^{\frac{1}{2}}, \\ \tilde{\phi}(\epsilon^2 t) &= \phi_0 - (\tfrac{5}{6}a_1^2 + \tfrac{3}{4}b_1)\, E_0\,\epsilon^2 t; \\ \tilde{A}(\epsilon^2 t) \equiv 0, \quad \tilde{B}(\epsilon^2 t) &= (2E_0)^{\frac{1}{2}}, \\ \tilde{\psi}(\epsilon^2 t) &= \psi_0 - (\tfrac{5}{6}a_2^2 + \tfrac{3}{4}b_3)\, E_0\epsilon^2 t + \tfrac{1}{2}\delta(\epsilon)\, t. \end{aligned}\right\} \tag{44}$$

Type II: $\tilde{X} = 0, 2\pi$:

$$\left.\begin{aligned} \tilde{A}^2(\epsilon^2 t) &= 2\frac{[10a_2(a_1+a_2) + 9(b_2-b_3)]\,E_0 + 6\delta(\epsilon)/\epsilon^2}{10(a_1+a_2)\,(3a_2-a_1) - 9(b_1-2b_2+b_3)}, \\ \tilde{B}^2(\epsilon^2 t) &= 2\frac{[10(a_1+a_2)\,(2a_2-a_1) - 9(b_1-b_2)]\,E_0 - 6\delta(\epsilon)/\epsilon^2}{10(a_1+a_2)\,(3a_2-a_1) - 9(b_1-2b_2+b_3)}, \\ \tilde{\phi}(\epsilon^2 t) &= \phi_0 - (\tfrac{5}{12}a_1^2 + \tfrac{3}{8}b_1)\,\tilde{A}^2\epsilon^2 t - (\tfrac{5}{12}a_1 a_2 + \tfrac{5}{6}a_2^2 + \tfrac{3}{8}b_2)\,\tilde{B}^2\epsilon^2 t, \\ \tilde{\psi}(\epsilon^2 t) &= \tilde{\phi}(\epsilon^2 t) - \tfrac{1}{2}\tilde{X}. \end{aligned}\right\} \tag{45}$$

Type III: $\tilde{X} = \pi, 3\pi$:

$$\left.\begin{aligned} \tilde{A}^2(\epsilon^2 t) &= 2\frac{[14a_2(a_1-a_2) + 3(b_2-3b_3)]\,E_0 + 6\delta(\epsilon)/\epsilon^2}{2(a_1-a_2)\,(9a_2-5a_1) - 3(3b_1-2b_2+3b_3)}, \\ \tilde{B}^2(\epsilon^2 t) &= 2\frac{[2(a_1-a_2)\,(2a_2-5a_1) - 3(3b_1-b_2)]\,E_0 - 6\delta(\epsilon)/\epsilon^2}{2(a_1-a_2)\,(9a_2-5a_1) - 3(3b_1-2b_2+3b_3)}, \\ \tilde{\phi}(\epsilon^2 t) &= \phi_0 - (\tfrac{5}{12}a_1^2 + \tfrac{3}{8}b_1)\,\tilde{A}^2\epsilon^2 t - (\tfrac{7}{12}a_1 a_2 - \tfrac{1}{6}a_2^2 + \tfrac{1}{8}b_2)\,\tilde{B}^2\epsilon^2 t, \\ \tilde{\psi}(\epsilon^2 t) &= \tilde{\phi}(\epsilon^2 t) - \tfrac{1}{2}\tilde{X}. \end{aligned}\right\} \tag{46}$$

In the expressions for $\tilde{\phi}(\epsilon^2 t)$ the appropriate values of $\tilde{A}^2$ and $\tilde{B}^2$ have to be substituted.

Expressions for the behaviour of the non-periodic orbits with time are cumbersome and will be omitted here except for the case of the global bifurcations.

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$\tilde{X} = 0, 2\pi$ (equations (31) hold). Eliminating $\tilde{B}$ and $\tilde{X}$ from equation (24*a*) with the aid of the integrals (26) and (33) and integration yields

$$\tilde{A}^2(\epsilon^2 t) = E_0^2 \pm (E_0^2 - \tfrac{1}{2}I_3)^{\frac{1}{2}} \sin\left((\tfrac{1}{6}a_1 a_2 - a_2^2 - \tfrac{1}{4}b_2)\,(2I_3)^{\frac{1}{2}}\epsilon^2 t + C\right). \tag{47}$$

The constant C is determined by the initial conditions. For $\tilde{B}^2(\epsilon^2 t)$ we get the same expression with $\pm$ replaced by $\mp$. Expressions for $\tilde{\phi}(\epsilon^2 t)$ and $\tilde{\psi}(\epsilon^2 t)$ can now be obtained by direct integration of equations (24*b*) and (24*d*). We omit these expressions here; for details in the case of the Hénon–Heiles problem see Verhulst (1977).

$\tilde{X} = \pi, 3\pi$ (equations (32) hold): Just replace I_3 by $-I_3$ in the expressions for the global bifurcations $\tilde{X} = 0, 2\pi$.

An interesting feature is displayed by these results, which is presumably characteristic for the flow near the global bifurcations. Consider again the $\tilde{A}^2$, $\tilde{X}$ phase plane in figure 3. The orbital curves are parametrized by the values of the integral I_3. At the centre points $\tilde{A}^2 = E_0$, $\tilde{X} = \pi, 3\pi$ and $\tilde{A}^2 = E_0$, $\tilde{X} = 0, 2\pi$ the extremal values $2E_0^2$ and $-2E_0^2$ are respectively attained. As we move outward the value of I_3 decreases/increases monotonically until it reaches the value zero on the location of global bifurcation $\tilde{X} = 0/\pi, 3\pi$. This means that (cf. equation (47)) the periods of the orbits become arbitrarily long on approaching the global bifurcations. To put it differently, if $I_3 = O(\epsilon)$, the time-scale of validity of the approximations $1/\epsilon^2$ is not long enough to describe the characteristics of the flow near the global bifurcations. This is another indication (cf. §9.3) that we need higher-order approximations to describe this phenomenon more adequately.

11. The resonance cases $n = 2$ and $n = 3$

The main resonances left over to be treated are the cases with resonance parameter ω near 2 and 3. In accordance with §5 we assume again $\delta(\epsilon) = O(\epsilon^2)$. Introduction of the modified Birkhoff transformation (12) into equations (10) and averaging produces after lengthy calculations equations for the first asymptotic approximations of the amplitudes and phases ($\tau = \epsilon^2 t$)

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tau} = O(\epsilon), \quad \frac{\mathrm{d}\tilde{B}}{\mathrm{d}\tau} = O(\epsilon), \tag{48\,a, b}$$

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau} = -(\tfrac{5}{12}a_1^2 + \tfrac{3}{8}b_1)\,\tilde{A}^2 - \left(\tfrac{1}{2}a_1 a_2 + \frac{a_2^2}{4n^2-1} + \tfrac{1}{4}b_2\right)\tilde{B}^2 + O(\epsilon), \tag{48\,c}$$

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tau} = -\left(\frac{a_1 a_2}{2n} + \frac{a_2^2}{n(4n^2-1)} + \frac{b_2}{4n}\right)\tilde{A}^2 - \left(\frac{8n^2-3}{4n(4n^2-1)}\,a_2^2 + \frac{3}{8n}b_3\right)\tilde{B}^2 + \frac{n}{2}\frac{\delta(\epsilon)}{\epsilon^2} + O(\epsilon) \tag{48\,d}$$

in which $n = 2$ or 3. It is surprising that in contrast with the main resonances at $n = \frac{1}{2}$ or $n = 1$ the resonances $n = 2$ and $n = 3$ exhibit for all values of the parameters $a_1, \ldots, b_3, \delta(\epsilon)$ degenerate behaviour, i.e. the amplitudes $A(t)$ and $B(t)$ are approximated by their initial values within error $O(\epsilon)$ on the time scale $1/\epsilon^2$ (§5). To give a complete description of the phase-flow we need approximations on a longer time-scale than $1/\epsilon^2$. Again, this subject falls within the scope of higher-order resonances and will be treated in a subsequent paper. At this stage we can calculate the variation with time of $\tilde{\phi}(\epsilon^2 t)$ and $\tilde{\psi}(\epsilon^2 t)$ by replacing $\tilde{A}$ and $\tilde{B}$ in equations (48*c, d*) by $A(0)$ and $B(0)$ and integrating. The resulting $\tilde{x}$ and $\tilde{z}$ constitute approximations of $x(t)$ and $z(t)$ with error $O(\epsilon)$ on the time scale $1/\epsilon^2$. Furthermore, it is evident that if $n = 2$ or $n = 3$

(*a*) Two independent approximate integrals exist which correspond to the respective energies

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in each of the two degrees of freedom. No exchange of energy between the two degrees of freedom takes place on the time scale $1/\epsilon^2$.

(*b*) The two normal modes (periodic) solutions $\tilde{A} = 0$ ($z, \dot{z}$ degree of freedom) and $\tilde{B} = 0$ ($x, \dot{x}$ degree of freedom) do both exist. It is easy to verify that this case presents no small denominator problems (cf. §7).

The degenerate behaviour at the main resonances $n = 2, 3$ is a consequence of the assumption of discrete symmetry in z of the potential (7) (or Hamiltonian (6)). In modelling actual physical situations one would consider the stability of such discrete symmetric systems by admitting small deviations from symmetry. In the example of a rotating galaxy discrete symmetry of the potential implies a mass distribution symmetric with respect to the galactic plane. A small deviation from discrete symmetry could be forced by a slightly asymmetric density distribution with regard to the galactic plane or by the attraction of a neighbouring galaxy. We consider the effect of such deviations by adding to the potential $U_3(x, z^2)$, given by equation (7), the asymmetric part

$$U_{3a} = -(a_3 x^2 z + \tfrac{1}{3} a_4 z^3) - (b_4 x^3 z + b_5 x z^3). \tag{49}$$

The equations of motion (8) become after rescaling

$$\left.\begin{aligned} \ddot{x} + x &= \epsilon(a_1 x^2 + a_2 z^2 + 2a_3 xz) + \epsilon^2(b_1 x^3 + b_2 x z^2 + 3b_4 x^2 z + b_5 z^3) + O(\epsilon^3), \\ \ddot{z} + \omega^2 z &= \epsilon(2a_2 xz + a_3 x^2 + a_4 z^2) + \epsilon^2(b_2 x^2 z + b_3 z^3 + b_4 x^3 + 3b_5 x z^2) + O(\epsilon^3). \end{aligned}\right\} \tag{50}$$

We repeat the procedure of §§4 and 5 with $n = 2$ and 3, without giving all the details.

$n = 2$.

The averaged equations for the approximate amplitudes and phases become

$$\left.\begin{aligned} \frac{d\tilde{A}}{d\tau} &= -\frac{a_3}{2}\tilde{A}\tilde{B}\sin\tilde{X}, \quad \frac{d\tilde{B}}{d\tau} = \frac{a_3}{8}\tilde{A}^2\sin\tilde{X}, \\ \frac{d\tilde{\phi}}{d\tau} &= -\frac{a_3}{2}\tilde{B}\cos\tilde{X}, \quad \frac{d\tilde{\psi}}{d\tau} = \frac{\delta(\epsilon)}{\epsilon} - \frac{a_3}{8}\frac{\tilde{A}^2}{\tilde{B}}\cos\tilde{X}, \end{aligned}\right\} \tag{51}$$

in which $\tau = \epsilon t$, $X = 2\phi - \psi$; we have omitted the $O(\epsilon^2)$ terms. Analysis of equations (51) gives the same type of results as obtained in the case $n = \frac{1}{2}$ (§§6 and 7). Again an integral independent of the energy exists which causes various local bifurcation phenomena. One could look at the case in which the asymmetric part ϕ_a is of smaller order than the symmetric part, for instance by putting in (49) $a_3 = \epsilon\bar{a}_3$, $a_4 = \epsilon\bar{a}_4$, etc., where the constants $\bar{a}_3, \bar{a}_4, \ldots$ are independent of ϵ. In this case one finds similar phenomena as displayed by equations (51) (we have to put $\tau = \epsilon^2 t$ and some terms have to be added in the averaged equations). This is a consequence of the fact that first-order averaging in the case $n = 2$ yields complete degeneracy, i.e. if $\delta(\epsilon) = O(\epsilon^2)$ all $O(\epsilon)$ terms are zero in the equations for the approximate amplitudes and phases.

$n = 3$.

We will not give a complete discussion of the phenomena associated with equations (50); it suffices here to consider the case $a_3 = a_4 = 0$. The averaged equations for the approximate amplitudes and phases become after applying modified Birkhoff transformation ($\tau = \epsilon^2 t$, $X = 3\phi - \psi$):

$$\frac{d\tilde{A}}{d\tau} = -\tfrac{3}{8} b_4 \tilde{A}^2 \tilde{B}\sin\tilde{X} + O(\epsilon), \quad \frac{d\tilde{B}}{d\tau} = \frac{b_4}{24}\tilde{A}^3\sin\tilde{X} + O(\epsilon). \tag{52a, b}$$

The equations for $d\tilde{\phi}/d\tau$ and $d\tilde{\psi}/d\tau$ are obtained by adding to equations (48*c*, *d*) for the case $n = 3$ the terms $-\frac{3}{8}b_4\tilde{A}\tilde{B}\cos\tilde{X}$ and $-\frac{1}{24}b_4(\tilde{A}^3/\tilde{B})\cos\tilde{X}$ respectively.

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The analysis of these equations for the case $n = 3$ runs along the same lines as the analysis for the case $n = 1$ (§§8–10). An integral independent of the energy exists which causes a number of local bifurcation phenomena. If one looks at the case in which the asymmetric part U_{3a} is of smaller order than the symmetric part (putting $a_3 = \epsilon\bar{a}_3, \dots, b_5 = \epsilon\bar{b}_5$ where the constants $\bar{a}_3, \dots, \bar{b}_5$ are independent of ϵ) equations (48 a, d) remain unchanged in the case $n = 3$ and again the two independent approximate integrals correspond with the respective energies in each of the two degrees of freedom.

12. The catastrophe set at the main resonances; structural stability

Let U be an open subset of $\mathbb{R}^4$ and $0 \in U$. We will consider functions (Hamiltonians) of the form (6) but we drop the assumption of analyticity. Consider instead the set $J^k(U, \mathbb{R})$ (functions with partial derivatives up to order k) of C^∞ functions of the form (6). The 4-jet of such functions is given by (6) and (7). We call this function space $H^k_s(U, \mathbb{R})$; two elements of this space will be called δ-close if their k partial derivatives with respect to U are δ-close in the sup-norm. To include perturbations which are not discrete symmetric with respect to z, we define a related function space as follows. Consider the Hamiltonian functions with non-degenerate critical point $0 \in \mathbb{R}^4$ of the form

$$h = \tfrac{1}{2}(\dot{x}^2 + \dot{z}^2) + U_4(x, z).$$

Again we restrict to $h \in J^k(U, \mathbb{R}) \cap C^\infty(U, \mathbb{R})$. We call this space $H^k(U, \mathbb{R})$ and we shall apply to it the same metric. Clearly $H^k_s(U, \mathbb{R}) \subset H^k(U, \mathbb{R})$. We consider these function spaces at each of the main resonances and we associate them with parameter spaces P^m which are isomorphic with $\mathbb{R}^m$; the parameters are generated by the k-jet of $U_3(x, z^2)$ in (7) or for $H^k(U, \mathbb{R})$ by (7) and (49). For instance, at the main resonance $n = 1$, $k = 4$, $m = 6$. In §§7.1 and 9.1 we found open sets $E \subset P^m$ corresponding with potentials $U_3(x, z^2)$ for which type II or type III bifurcations exist. The boundary set ∂E includes in §9.1 the global bifurcations with respect to the energy. The map $\partial E \to H^k_s(U, \mathbb{R})$ produces the catastrophe set C_E in $H^k_s(U, \mathbb{R})$. In §§7.2 and 9.3 we found open sets $S \subset P^m$ corresponding to potentials $U_3(x, z^2)$ for which the type I, II and III bifurcations are stable or unstable. The map $\partial S \to H^k_s(U, \mathbb{R})$ produces the catastrophe set C_S in $H^k_s(U, \mathbb{R})$. Finally we found in §§6.1 and 8.2 conditions of degeneracy which takes place at a set $D \subset P^m$. The map $D \to H^k_s(U, \mathbb{R})$ produces the catastrophe set C_D in $H^k_s(U, \mathbb{R})$.

Remark. In catastrophe theory the standard terminology would be to call the set $\partial E \cup \partial S \cup D \subset P^m$ the catastrophe set and the set induced by the mapping of the catastrophe set into $H^k_s(U, \mathbb{R})$ the bifurcation set. We refrain from adopting this terminology to avoid introducing a concept of bifurcation with yet another meaning. If a Hamiltonian $h \in H^k_s(U, \mathbb{R})$ is not contained in the catastrophe set $C_E \cup C_S \cup C_D$ we call this Hamiltonian (and the corresponding potential) structurally stable in $H^k_s(U, \mathbb{R})$. An analogous definition of structural stability applies to Hamiltonians in $H^k(U, \mathbb{R})$.

We summarize the results relevant to structural stability by defining the sets ∂E, ∂S and D in P^m for each of the main resonances. As explained earlier the results depend on the energy E_0.

$\underline{n = \tfrac{1}{2}}$: $k = 3$, $m = 3$; parameters $a_1, a_2, \delta(\epsilon)$ $(= O(\epsilon))$

set		reference section
D	$a_2 = 0$	6.1
∂E	$\delta(\epsilon) = -\tfrac{1}{2}\sqrt{2}\, a_2 \epsilon E_0$ $\delta(\epsilon) = \tfrac{1}{2}\sqrt{2}\, a_2 \epsilon E_0$	7.1
∂S	$\partial S = \partial E$	7.2

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$n = 1$: $k = 4, m = 6$, parameters $a_1, a_2, b_1, b_2, b_3, \delta(\epsilon)\ (= O(\epsilon^2))$

set		reference section
D	$b_2 = \frac{2}{3}a_2(a_1-6a_2)$	8.2
∂E	$\frac{10a_2(a_1+a_2)+9(b_2-b_3)+6\delta(\epsilon)/(E_0\epsilon^2)}{10(a_1+a_2)(3a_2-a_1)-9(b_1-2b_2+b_3)} = 0$ and 1	9.1
	$\frac{14a_2(a_1-a_2)+3(b_2-3b_3)+6\delta(\epsilon)/(E_0\epsilon^2)}{2(a_1-a_2)(9a_2-5a_1)-3(3b_1-2b_2+3b_3)} = 0$ and 1	
∂S	$b_2 = \frac{2}{3}a_2(a_1-6a_2)$; $10(a_1+a_2)(a_1-3a_2)+9(b_1-2b_2+b_3) = 0$ restricted to $E \subset P^6$.	9.3
	$2(a_1-a_2)(5a_1-9a_2)+3(3b_1-2b_2+3b_3) = 0$ restricted to $E \subset P^6$.	
	$\left\lvert \frac{2(4a_2^2+6a_1a_2-5a_1^2)-3(3b_1-2b_2)-6\delta(\epsilon)/(E_0\epsilon^2)}{2a_2(a_1-6a_2)-3b_2} \right\rvert = 1$	
	$\left\lvert \frac{2a_2(6a_1-a_2)+3(2b_2-3b_3)+6\delta(\epsilon)/(E_0\epsilon^2)}{2a_2(a_1-6a_2)-3b_2} \right\rvert = 1$	

$n = 2$ and 3

To study the structural stability in $H_s^k(U, \mathbb{R})$ we need results on higher-order resonances which will be given in a subsequent paper. It is however clear from § 12 that the potential (6) is structurally unstable in $H^k(U, \mathbb{R})$ at resonances $n = 2$ and 3.

13. Applications and conclusions

13.1. *Model galaxies*

In constructing models for axi-symmetric rotating galaxies which are symmetric with respect to the galactic plane, structurally stable potentials at the main resonances $n = \frac{1}{2}$ and $n = 1$ can be obtained (§ 12). One must choose the potentials outside the indicated catastrophe set. The terms in the Taylor expansion of the potential up to (and including) degree 3 in the case of $n = \frac{1}{2}$ and up to degree 4 in the case of $n = 1$ determine the structural stability completely. The situation is different at the main resonances $n = 2$ and $n = 3$, where we have degeneration of the resonances (§ 11). Axi-symmetric rotating galaxies are not structurally stable at these resonances if deviations from symmetry with respect to the galactic plane are admitted.

13.2. *The 'third integral of the galaxy'*

A third integral of the galaxy (isolating and independent of the energy and angular momentum) for axi-symmetric rotating galaxies which are symmetric with respect to the galactic plane exists in an asymptotic sense at and in the neighbourhood of the main resonances $n = \frac{1}{2}$ and $n = 1$ if the potential is chosen outside the catastrophe set C_D. The asymptotic is not only formal but based on rigorously established estimates (§§ 6.2 and 8.2). In the neighbourhood of the resonances $n = 2$ and $n = 3$ we have angular momentum as an exact integral and two independent approximate integrals corresponding with the respective energies in each of the two degrees of freedom (§ 11). If we admit model galaxies with deviations from symmetry with respect to the galactic plane we have three integrals, angular momentum, energy and an approximate integral producing exchange of energy between the two degrees of freedom.

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13.3. *Density waves*

The local bifurcations found in §§7.1 and 9.1 correspond with continuous sets of periodic solutions if we do not keep the energy constant. It can be shown that these sets of periodic solutions correspond with axi-symmetric density waves by calculating the density response from the Poisson equation (2). These calculations are easy to perform and the details have been left to the reader.

13.4. *Classical examples of potentials*

In the literature a number of axially symmetric potentials with isolating integrals can be found and we shall show how some of them fit into our classification. We shall refer to table I given by Lynden-Bell (1962); we use the notation employed there except that we put $R^2 = r^2+z^2$ (r is the coordinate used in a cylindrical frame of reference). Except where explicitly stated otherwise, structural stability is considered in $H^k_s(U, \mathbb{R})$.

(i) $\psi = [\zeta(\lambda)-\eta(\mu)]/(\lambda-\mu)$ and the approximation called Eddington's potential: ζ and η can be chosen so that the potential is structurally stable at $n = \frac{1}{2}$ and $n = 1$. Model galaxies with such a potential, which can be expanded in the form (7), should only be used near $n = 2$ and $n = 3$ if $\zeta(\lambda)$ and $\eta(\mu)$ have been chosen so that the energy in each of the two degrees of freedom separately is conserved.

(ii) $\psi = \zeta(r, z)$: this is the general potential studied in this paper.

(iii) $\psi = \zeta(R)$: the spherically symmetric potential is for each choice of ζ (sufficiently smooth) structurally unstable at $n = 1$ and $n = \frac{1}{2}$. The structural instability in $H^k(U, \mathbb{R})$ at $n = 2$ and 3 suggests that we have to avoid choices of $\zeta(R)$ which can be locally approximated by $AR^{-\frac{7}{4}}$ ($n = 2$) or $AR^{-\frac{17}{9}}$ ($n = 3$).

(iv) $\psi = \zeta(R)+R^{-2}\eta(\theta)$; we have the same conclusions as in *i*.

(v) $\psi = \zeta(r)+\eta(z)$; structurally unstable.

(vi) $\psi = AR^{-1}$; structurally unstable ($\omega = n = 1$).

(vii) $\psi = Ar^{-1}+\zeta(z)$; structurally unstable.

(viii) $\psi = Ar^2+\zeta(z)$; structurally unstable.

(ix) $\psi = -A(l^2r^2+n^2z^2)$; structurally unstable.

(x) $\psi = -AR^2$; structurally unstable ($\omega = n = \frac{1}{2}$).

13.5. *The Hénon–Heiles problem*

In the problem discussed by Hénon & Heiles (1964) we have $n = 1$, $\delta(\epsilon) = 0$, $a_1 = 1$, $a_2 = -1$; b_1, b_2, b_3 and all other coefficients in the expansion for the potential (7) are zero. An extensive discussion of this problem is given by Verhulst (1977). We summarize here the additional information obtained from §9. The Hénon–Heiles problem provides an example of a potential described by Braun's one-parameter family of bifurcations (§9.4). Clearly $\lambda = -\frac{1}{3}$. The potential is structurally unstable as at $\lambda = -\frac{1}{3}$ we find a global bifurcation; moreover, the two normal modes and the type II bifucations change their stability characteristics at this point. The periodic solutions are:

(I) Normal mode solutions

Approximate solution at $x = \dot{x} = 0$:

$$\tilde{z} = (2E_0)^{\frac{1}{2}} \cos(\psi_0+t-\tfrac{5}{6}E_0\,\epsilon^2 t),$$
$$\dot{\tilde{z}} = -(2E_0)^{\frac{1}{2}} \sin(\psi_0+t-\tfrac{5}{6}E_0\,\epsilon^2 t).$$

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Approximate solution at $z = \dot{z} = 0$:

$$\tilde{x} = (2E_0)^{\frac{1}{2}} \cos(\phi_0 + t - \tfrac{5}{6}E_0 \epsilon^2 t),$$
$$\dot{\tilde{x}} = -(2E_0)^{\frac{1}{2}} \sin(\phi_0 + t - \tfrac{5}{6}E_0 \epsilon^2 t).$$

Exact solution at $z = \dot{z} = 0$ described by

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{3}x^3 = h.$$

(II) Two exact solutions at $X = 0, 2\pi$ given by

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}\dot{x}^2 + \tfrac{2}{3}x^3 = \tfrac{1}{4}h \quad \text{and} \quad z^2 = 3x^2.$$

Two global bifurcations at $\tilde{X} = 0, 2\pi$.
(III) Two periodic solutions at $\tilde{X} = \pi, 3\pi$ given by the approximations ($p = \pi/2$ or $3\pi/2$)

$$\tilde{x} = E_0^{\frac{1}{2}} \cos(\phi_0 + t + \tfrac{1}{3}E_0 \epsilon^2 t), \qquad \tilde{z} = E_0^{\frac{1}{2}} \cos(\phi_0 - p + t + \tfrac{1}{3}E_0 \epsilon^2 t),$$
$$\dot{\tilde{x}} = -E_0^{\frac{1}{2}} \sin(\phi_0 + t + \tfrac{1}{3}E_0 \epsilon^2 t), \quad \dot{\tilde{z}} = -E_0^{\frac{1}{2}} \sin(\phi_0 - p + t + \tfrac{1}{3}E_0 \epsilon^2 t).$$

Hénon and Heiles constructed a surface of section in the $x, \dot{x}$ plane ($z = 0, \dot{z} > 0$). The boundary of this surface of section is given by the exact normal mode at $z = \dot{z} = 0$. Fixed points are found from the approximate normal mode at $x = \dot{x} = 0$, the two type II and the two type III periodic solutions. We find

type II (exact)	type III (approximately)
$x = 0$	$x = \pm\sqrt{h}$
$\dot{x} = \pm\sqrt{(h/2)}$	$\dot{x} = 0.$

We conclude with two remarks

(1) The fixed points associated with the type II and type III solutions can be recognized in the pictures given by Hénon and Heiles for $h = 0.08333$ and $h = 0.12500$. The fixed points associated with the exact type II solutions exist up to the critical energy $h = \frac{1}{6}$. At this point the energy manifold bifurcates and the solutions associated with type II bifurcations for $h < \frac{1}{6}$ escape to infinity for $h > \frac{1}{6}$.

(2) In § 9.3 we remarked that our discussion of the global bifurcations is not complete as in this discussion of existence and stability the global bifurcations with respect to the energy correspond with critical points which are degenerate in the sense of Morse theory. In this context it is interesting to note that the type II exact solutions (which are found at the locations $\tilde{X} = 0, 2\pi$ of the global bifurcations) are stable in Hénon and Heiles' numerical study. The higher-order normal forms clearly stabilize the solutions in this case.

(*f*) *A remark on Contopoulos' third integral*

An expansion for the 'third integral of the galaxy' has been given by Contopoulos (1967) for the potential (7) with $a_1 = b_1 = b_2 = b_3 = 0$; no proof of convergence or asymptotic character of Contopoulos' results is known but there is good agreement with a number of numerical results. One can show that our approximate first integral (28*a*) is equivalent to Contopoulos' integral found by Contopoulos & Moutsoulas (1966) at $n = 1$ to a certain order of expansion. This proves the asymptotic validity of Contopoulos' integral for that case. A more complete comparison of Contopoulos' analytical results and ours should wait until we have discussed the higher order resonances in a subsequent paper.

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